Redshift Space Distortions

Overview

It is important to realize that our "third dimension" in cosmology is not radial distance but redshift. The two are related by the Hubble expansion but also affected by *peculiar* velocities. On small scales random motions within e.g. a cluster of galaxies will cause particles at the same distance to have slightly different redshifts. This elongates structures along the line of sight and leads to the so-called fingers-of-God effect: structures have a tendency to point toward the observer. On very large scales the opposite happens. Objects fall in towards overdense regions. This makes objects between us and the overdensity appear to be further away and objects on the other side of the overdensity appear closer. The net effect is to enhance the overdensity rather than smear it out. These effects are known as *redshift space distortions* – you can find a review in astro-ph/9708102 (specifically §§1, 2 5 & 6).

The Kaiser factor

The former effect is fairly difficult to calculate, but the latter is a simple exercise in linear perturbation theory. You may wish to look at Nick Kaiser's original paper on this subject (MNRAS, 227, 1, 1987). Denote the redshift space coordinate as $\vec{s}$ and the real space coordinate as $\vec{r}$. If $\vec{v}(\vec{r})$ is the peculiar velocity in units of the Hubble constant then

$$\vec{s} = \vec{r}\left(1 + \frac{u(r)}{r}\right) \tag{1}$$

where $u = \hat{r} \cdot \vec{v}(\vec{r})$ and we have assumed $v(0) = 0$. Assuming that the object is so distant[1] that $kr \gg 1$ the Jacobian between $\vec{s}$ and $\vec{r}$ is

$$d^3s = \left(1 + \frac{du}{dr}\right) d^3r \tag{2}$$

Number density conservation requires $\delta_s d^3s = \delta_r d^3r$ for plane-wave perturbation δ. Using the linear theory relation $\dot{\delta} = -ikv$ we have

$$\frac{du}{dr} = -\mu^2 f(\Omega)\delta \tag{3}$$

[1]This is not a good approximation for wide-angle surveys, see Papai & Szapudi 0802.2940 for more details.

where $\mu = \hat{r} \cdot \hat{k}$ and $f(\Omega) \equiv d\log\delta/d\log a \approx \Omega^{0.6}$. Putting it all together we have

$$\delta_s = \delta_r \left(1 + f\mu^2\right) \tag{4}$$

If we presume that the galaxy fluctuation is biased with respect to the mass by $\delta_{\rm gal} = b\delta_{\rm mass}$ then

$$\Delta^2_{\rm gal}(k,\mu) = \left(1 + \beta\mu^2\right)^2 \Delta^2_{\rm mass}(k) \tag{5}$$

where $\beta \equiv f(\Omega)/b$. Note that β is not really a cosmological parameter like Ω as it is only defined in the context of the most simple linear biasing model (which is almost certainly wrong). However it is often referred to by people who study velocity fields because $f(\Omega)/b$ relates the velocity field to the density field in this linear bias model.

The μ dependence of $\Delta^2_{\rm gal}(k,\mu)$ can be expanded out in Legendre polynomials of order 0, 2, 4. The coefficient of P_0 is $1+\frac{2}{3}\beta+\frac{1}{5}\beta^2$ which is the amount by which clustering is enhanced on large-scales by redshift space distortions. Notice that for $\Omega \sim b \sim 1$ this can by a significant effect! Formulae for the higher moments can be easily derived – you can find the expressions in real space in Hamilton's review (astro-ph/9708102) while in Fourier space:

$$\begin{aligned} \frac{\Delta^2_{\rm red}(k,\mu)}{\Delta^2_{\rm real}(k)} &= \left[\left(1 + \frac{2}{3}\beta + \frac{1}{5}\beta^2\right) P_0(\mu) \right. && (6) \\ &+ \left(\frac{4}{3}\beta + \frac{4}{7}\beta^2\right) P_2(\mu) && (7) \\ &+ \left. \frac{8}{35}\beta^2 P_4(\mu)\right] && (8) \end{aligned}$$

For quite some time it was believed that the ratio of the quadrupole to monopole term would be a good way to measure β – see Berlind, Narayanan & Weinberg (2001; ApJ 549, 688) for a summary of past work and a discussion of issues with this approach.

In deriving Eq. (5) we assumed the distant observer approximation. This is not necessary. In moving to wide fields it makes sense to make a radial-angular decomposition, since redshift space distortions involve only radial remapping. Thus one expands

$$\rho_{\ell mn}(s) = c_{\ell n} \int d^3s\ \rho(\vec{s}) j_\ell(k_{\ell n} s) Y^*_{\ell m}(\theta,\phi) \tag{9}$$

where j_ℓ are the spherical Bessel functions and $Y_{\ell m}$ the spherical harmonics. Redshift space distortions then become relations between the $\rho_{\ell mn}$. For example

$$j_\ell\left(k_{\ell n}s\right) \simeq j_\ell\left(k_{\ell n}r\right) + v(r)\frac{d}{dr}j_\ell\left(k_{\ell n}r\right) + \cdots \tag{10}$$

and the derivative can be written, using recurrence relations, in terms of the $j_{\ell+1}$ and $j_{\ell-1}$. If one is going to drop the distant observer approximation, one also needs to worry about the fx_jv_j/x^2 term – see Papai & Szapudi.

Numerous numerical simulations have shown that the convergence to the Kaiser factor is very slow. Thus for precision cosmology the Kaiser effect represents a limit which is never valid for real surveys. There is even a claim (Scoccimarro 2004; PRD 70, 083007) that Kaiser is invalid even for $k \to 0$.

Fingers of God

To deal with small-scale velocities formally requires a model for the velocity field and its correlations with density. This is quite difficult. For the density we have the exact expression

$$\delta_D(k) + \delta_s(k) = \int \frac{d^3x}{(2\pi)^3}\; e^{-ik\cdot x} e^{ifk_z u_z(x)}\left[1+\delta(x)\right] \tag{11}$$

where δ_D is a Dirac δ-function. The power spectrum is thus

$$\delta_D(k) + P_s(k) = \int \frac{d^3r}{(2\pi)^3}\; e^{-ik\cdot x}\left\langle e^{ifk_z\Delta u_z(x)}\left[1+\delta(x)\right]\left[1+\delta(x')\right]\right\rangle \tag{12}$$

where $r = x - x'$. Unfortunately this involves knowing all of the density and velocity correlations. There is a similar formula for the correlation function:

$$1 + \xi_s(s_{||}, s_\perp) = \int dr_{||}\;\left[1+\xi(r)\right]P(r_{||} - s_{||}, r) \tag{13}$$

where P is the velocity distribution function which depends on r. In general it is almost impossible to make rigorous progress here. If we neglect this r dependence and assume isotropic velocity dispersion we get the "streaming model" first introduced by Peebles and used for many years: ξ_s is just a convolution of ξ_r and a velocity PDF. In Fourier space the convolution is a multiplication so Δ_s^2 becomes Δ_r^2 times the FT of the velocity model, usually a Gaussian or an exponential. This is an extremely simple, though not rigorously justified, model.

Modelling the distortions

We saw above that frequently one uses a simplified model for the redshift space distortions. For example you could multiply the Kaiser factor by a small-scale factor which is either a Gaussian or an exponential. In Fourier space the latter leads to the "dispersion model" (see for example Peacock & Dodds, 1994, MNRAS 267, 1020)

$$\Delta^2_{\rm red}(k,\mu) = \Delta^2(k)\frac{(1+\beta\mu^2)^2}{1+k^2\mu^2\sigma^2/2} \tag{14}$$

where β and σ are parameters to be fit to the data. Expressions for the moments can be found in Cole, Fisher & Weinberg (1995; MNRAS, 275, 512). A generalization to the halo model, and a description of why an exponential model is better than a Gaussian, can be found in (MNRAS, 321, 1 (2001); MNRAS 325, 1359 (2001)).

Beyond dispersion or streaming models one needs to turn to numerical simulations. Some work in this direction has been reported in e.g. Hatton & Cole (1999; MNRAS, 310, 113) who find a fit to the quadrupole-to-monopole ratio $Q = Q_{\rm lin}(1-x^{1.22})$ where $x = k/k_1$ and k_1 is a free parameter akin to σ above.