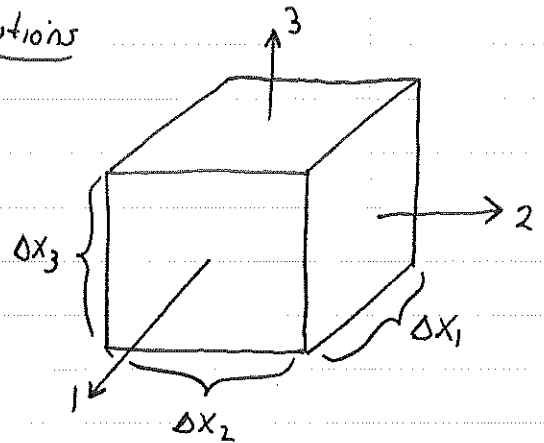


Assignment 1 Solutions

① Consider the small box shown. The coordinate origin is at the center of the box. What are the surface torques about axis 3?



Consider the face pierced by axis 1, closest to us. There is a force in the 2-direction acting on this face: $F_2 = \sigma_{21} \Delta x_2 \Delta x_3$. Since the face is a distance $\Delta x_1/2$ from the 3-axis, F_2 exerts a torque $\tau = \sigma_{21} \Delta x_1 \Delta x_2 \Delta x_3 / 2$.

There is another force acting on the face pierced by axis 1 that is farther from us. Since the box is not accelerating, this force is equal and opposite to the first one. [The force on each face scales as Δx^2 , while the mass of fluid scales as Δx^3 . Thus, the opposite forces must cancel to avoid an infinite acceleration.] The total torque acting in the 3-direction, arising from these two forces is

$$\tau_{\text{tot}} = \sigma_{21} \Delta x_1 \Delta x_2 \Delta x_3$$

Next consider the forces pierced by the 2-axis. The right face experiences a force in the 1-direction of $F_1 = \sigma_{12} \Delta x_1 \Delta x_3$, and contributes a torque about the 3-axis of $\tau = \sigma_{12} \Delta x_1 \Delta x_2 \Delta x_3 / 2$. Adding the contribution from the left face, and invoking force balance, gives a total torque $\tau = \sigma_{12} \Delta x_1 \Delta x_2 \Delta x_3$.

The faces pierced by the 3-axis exert no torques in the 3-direction. Since the box is not rotating about the 3-axis, the associated torque must be zero. [The torque scales as Δx^3 , while the moment of inertia scales as Δx^5 . So all torques must cancel to avoid an infinite angular acceleration.]

We have $\sum \tau = 0 = (\sigma_{21} - \sigma_{12}) \Delta x_1 \Delta x_2 \Delta x_3 \rightarrow \sigma_{12} = \sigma_{21}$

Analogous reasoning applied to the other axes leads to the conclusion:

$$\sigma_{ij} = \sigma_{ji}$$

② C & C Problem 3 -

(a) Streamlines are curves whose tangents, at any time, lie in the direction of the fluid velocity. In Cartesian coordinates, they are the level surfaces of functions $f(x, y, z)$. That is, they are traced by

$$f(x, y, z) = f_0,$$

where f_0 can be a function of time. Along these level surfaces,

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

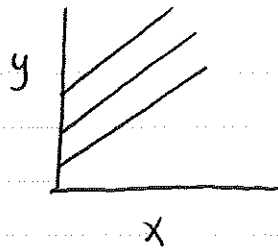
For the fluid velocity $\vec{u} = (u_x, u_y e^{-t/\tau}, 0)$, where u_x, u_y, τ are all constants, the motion lies entirely in the (x, y) plane.

We have
$$\frac{dy}{dx} = \frac{u_y}{u_x} e^{-t/\tau}$$

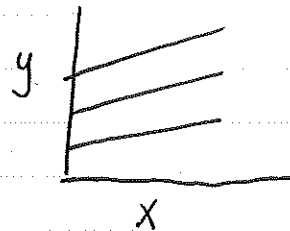
The slope of the streamlines is uniform (independent of x and y), but decreases with time.

Mathematically $f(x, y) = \frac{y - y_0}{x}$ $f_0 = (u_y/u_x) e^{-t/\tau}$

Here $y_0(x)$ is an arbitrary function of x that gives the value of y when $x=0$:



$t = t_1$



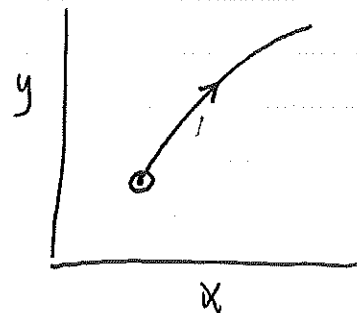
$t = t_2 > t_1$

(b) Particle paths are found from $\frac{d\vec{r}}{dt} = \vec{u}$. In our case,

$$\frac{dx}{dt} = u_x \quad \frac{dy}{dt} = u_y e^{-t/\tau}$$

Both x and y increase with time, but y is decelerating.

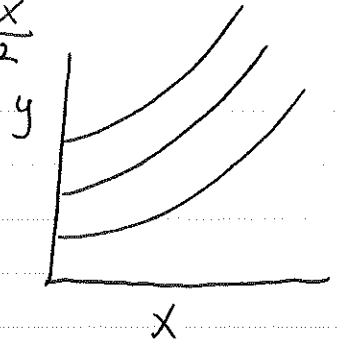
Integrating, $x = x_0 + u_x t$
 $y = y_0 + u_y \tau (1 - e^{-t/\tau})$



③ C & C Problem 5

(a) Streamlines are found from $\frac{dy}{dx} = \frac{u_y}{u_x} = \frac{1}{2/x} = \frac{x}{2}$

Integrating - $y = \frac{x^2}{4} + \text{const}$ (parabolas)



(b) In steady state, $\nabla \cdot (\rho \vec{u}) = 0$

Integrating over Z , and using $\Sigma \equiv \int \rho dz$,

$$\frac{\partial}{\partial x} (\Sigma u_x) + \frac{\partial}{\partial y} (\Sigma u_y) = 0 \quad \text{let } \Sigma(x,y) = \underline{X}(x) \underline{Y}(y)$$

$$\underline{Y} \frac{\partial}{\partial x} (\underline{X} u_x) + \underline{X} \frac{\partial}{\partial y} (\underline{Y} u_y) = 0 \rightarrow \underline{Y} \frac{\partial}{\partial x} \left(\frac{2\underline{X}}{x} \right) + \underline{X} \frac{\partial}{\partial y} (\underline{Y}) = 0$$

Dividing by $\underline{X}\underline{Y}$, $\frac{1}{\underline{X}} \frac{\partial}{\partial x} \left(\frac{2\underline{X}}{x} \right) + \frac{1}{\underline{Y}} \frac{\partial}{\partial y} \underline{Y} = 0$ Each term must be a constant; call this λ

$$\frac{1}{\underline{X}} \frac{d}{dx} \left(\frac{2\underline{X}}{x} \right) = \lambda \rightarrow \frac{d\underline{X}}{dx} = \underline{X} \left(\frac{1}{x} + \frac{\lambda x}{2} \right) \rightarrow \underline{X} = \underline{X}_0 x e^{\lambda x^2/4}$$

$$\frac{1}{\underline{Y}} \frac{d\underline{Y}}{dy} = -\lambda \rightarrow \underline{Y} = \underline{Y}_0 e^{-\lambda y} \rightarrow \Sigma = A x e^{\lambda x^2/4 - \lambda y}$$

where λ and A are constants

(c) The surface density of nuclei is Σ times

their number per unit mass. Since the latter decays as e^{-t} , we have

$$N = B x e^{\lambda \left(\frac{x^2}{4} - y \right) - t} \quad \text{But along a streamline, } \frac{x^2}{4} - y = \text{constant, so}$$

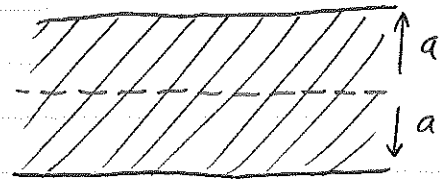
$$N = C x e^{-t} \quad \text{The time } t \text{ is that required to travel from } x_0 \text{ to } x.$$

$$N = D x e^{-\frac{x^2}{4}} \quad \text{Since } u_x = \frac{dx}{dt} = \frac{2}{x}, \text{ we have } t = \frac{x^2}{4} - \frac{x_0^2}{4}. \text{ Thus,}$$

$$\text{We want } 0 = \frac{dN}{dx} = D e^{-x^2/4} \left(1 - \frac{x^2}{2} \right) \rightarrow \boxed{X = \sqrt{2}}$$

If $x_0 > \sqrt{2}$, the concentration of nuclei declines immediately.

④ C & C Problem 8 -



$$(a) \quad 0 = -\frac{1}{\rho} \frac{dP}{dz} - \frac{d\psi}{dz} \quad (\rho = \text{const})$$

$$\frac{d^2\psi}{dz^2} = 4\pi G\rho \rightarrow \psi(z) = \psi_0 + \psi_1 z + 2\pi G\rho z^2 \quad \psi_1 \text{ and } \psi_2 \text{ are constants}$$

By symmetry $g(0) = -\frac{d\psi}{dz}(0) = 0 \rightarrow \psi_1 = 0$

Since $\psi(z)$ has an arbitrary additive constant, we can set $\psi_0 = 0$

$$\boxed{\psi(z) = 2\pi G\rho z^2}$$

$$\boxed{g = -\frac{\partial\psi}{\partial z} = -4\pi G\rho z}$$

From hydrostatic balance, $0 = -\frac{1}{\rho} \frac{dP}{dz} - 4\pi G\rho z$

$$\frac{dP}{dz} = -4\pi G\rho^2 z \rightarrow \boxed{P = P_{\text{ext}} + 2\pi G\rho^2 (a^2 - z^2)}$$

where P_{ext} is whatever external pressure is imposed on the slab.

(b) The motion of the star is given by $\ddot{z} = g = -4\pi G\rho z$

The solution, starting at rest at $z=a$, is $z(t) = a \cos \omega t$

$$\ddot{z} = -\omega^2 a \cos \omega t$$

$$= -4\pi G\rho a \cos \omega t$$

Thus $\omega^2 = 4\pi G\rho \rightarrow \omega = \sqrt{4\pi G\rho} = 2\pi/T$

The period $\boxed{T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{4\pi G\rho}} = \sqrt{\frac{\pi}{G\rho}}$

$$\boxed{V_{\text{max}} = \omega a = a \sqrt{4\pi G\rho}}$$

(c) $\rho = 10^{-18} \text{ kg m}^{-3} = 10^{-21} \text{ g cm}^{-3}$; $a = 10^{18} \text{ m} = 10^{20} \text{ cm}$

$$\boxed{T = \sqrt{\frac{\pi}{G\rho}} = 2.2 \times 10^{14} \text{ s} = 6.9 \times 10^6 \text{ yr}}$$

use $G = 6.67 \times 10^{-8} \text{ (cgs)}$

$$\boxed{V_{\text{max}} = \frac{2\pi a}{T} = 2.9 \times 10^6 \text{ cm s}^{-1} = 29 \text{ km s}^{-1}}$$