Ay 202
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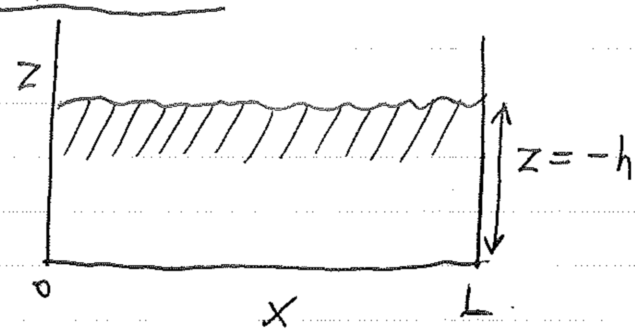
Assignment 2 Solutions



① $\varphi(x,z,t) = \varphi_0(z) f(x) \cos\omega t$
which obeys $\nabla^2 \varphi = 0$

(a) At $z=0$, $\frac{\partial^2\varphi}{\partial t^2} = -g\frac{\partial\varphi}{\partial z}$
(Free surface)
At $z=-h$, $\frac{\partial\varphi}{\partial z} = 0$ (no penetration of bottom)
At $z = 0, L$ $\frac{\partial\varphi}{\partial x} = 0$ (no penetration of sides)

(b) $\nabla^2\varphi = 0 \rightarrow 0 = \varphi_0''(z) f(x)\cos\omega t + \varphi_0(z) f''(x)\cos\omega t$

$$0 = \frac{\varphi_0''(z)}{\varphi_0(z)} + \frac{f''(x)}{f(x)} \quad \text{so that}$$

$$\left.\begin{aligned}\frac{\varphi_0''(z)}{\varphi_0(z)} = k^2 \quad \text{and} \quad \frac{f''(x)}{f(x)} = -k^2\end{aligned}\right\} \begin{aligned}\varphi_0(z) &= \varphi_1 e^{kz} + \varphi_2 e^{-kz}\\ f(x) &= f_1\cos kx + f_2 \sin kx\end{aligned}$$

$$\varphi = (\varphi_1 e^{kz} + \varphi_2 e^{-kz})(f_1\cos kx + f_2\sin kx)\cos\omega t$$

$$\frac{\partial\varphi}{\partial z}(z=-h) = k(\varphi_1 e^{-kh} - \varphi_2 e^{+kh})(f_1\cos kx + f_2\sin kx)\cos\omega t = 0$$

$$\rightarrow \varphi_2 = \varphi_1 e^{-2kh} \rightarrow \varphi = \varphi_1(e^{kz} + e^{-2kh}e^{-kz})\begin{pmatrix}f_1\cos kx +\\ f_2\sin kx\end{pmatrix}\cos\omega t$$

$$\frac{\partial\varphi}{\partial x}(x=0) = k\varphi_1(e^{kz} + e^{-2kh}e^{-kz}) f_2\cos\omega t = 0 \rightarrow f_2 = 0$$

$$\frac{\partial\varphi}{\partial x}(x=L) = k\varphi_1(e^{kz} + e^{-2kh}e^{-kz})(-f_1\sin kL) = 0 \rightarrow kL = 0, \pi, 2\pi, \text{etc}$$

$$\boxed{\varphi = \varphi_1 f_1 (e^{kz} + e^{-2kh}e^{-kz})\cos kx\cos\omega t}, \text{ where } k = 0, \frac{\pi}{L}, \frac{2\pi}{L}, \text{etc}$$

$$\frac{\partial^2\varphi}{\partial t^2} = -\omega^2\varphi_1 f_1(e^{kz} + e^{-2kh}e^{-kz})\cos kx\cos\omega t$$

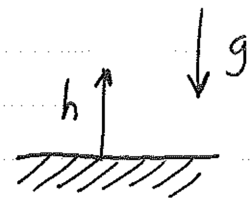
To satisfy the 1st boundary condition, $-\omega^2\varphi_1 f_1(1+e^{-2kh})\cos kx\cos\omega t =$
$-gk\varphi_1 f_1(1-e^{-2kh})\cos kx\cos\omega t$

Thus, $\boxed{\omega^2 = gk\frac{1-e^{-2kh}}{1+e^{-2kh}}}$ as before

(c) For the fundamental mode, $k = \frac{\pi}{L}$, and $\boxed{P_0 = \frac{2\pi}{\sqrt{gk}}\left(\frac{1+e^{-2kh}}{1-e^{-2kh}}\right)^{1/2}}$

(d) For a bathtub, $L \simeq 1.5$ m and $h \simeq 0.5$ m
Thus $P_0 = 1.6$ seconds
which is reasonable.

② (a) Hydrostatic balance: $\frac{1}{\rho}\frac{dP}{dh} = -g$



We also have: $P = k\rho^{\gamma}$

$$P = \frac{\rho R T}{\mu}$$

Thus $\gamma k \rho^{\gamma-2}\frac{d\rho}{dh} = -g \longrightarrow \rho^{\gamma-2}\,d\rho = \frac{-g}{\gamma k}\,dh$

$$\int_{\rho_0}^{\rho} \rho^{\gamma-2}\,d\rho = \frac{-gh}{\gamma k} \longrightarrow \frac{1}{\gamma-1}\left[\rho^{\gamma-1} - \rho_0^{\gamma-1}\right] = \frac{-gh}{\gamma k}$$

$$\rho^{\gamma-1} - \rho_0^{\gamma-1} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{gh\rho_0^{\gamma}}{P_0} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{gh\rho_0^{\gamma-1}}{P_0/\rho_0} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{\mu g h}{R T_0}\rho_0^{\gamma-1}$$

$$\boxed{\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = 1 - \left(\frac{\gamma-1}{\gamma}\right)\frac{\mu g h}{R T_0}}$$

(b) From the two expressions for pressure, $k\rho^{\gamma} = \rho\frac{RT}{\mu} \rightarrow \rho^{\gamma-1} = \frac{RT}{\mu k}$

Thus $\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \frac{T}{T_0} = 1 - \left(\frac{\gamma-1}{\gamma}\right)\frac{\mu g h}{R T_0}$ Differentiating,

$$\boxed{\frac{dT}{dh} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{\mu g}{R}}$$

For the Earth's atmosphere,

$g = 980$ cm s^{-1}

$R = 8.31 \times 10^{7}$ erg deg^{-1}

$\mu = 29$

$\gamma = 7/5$ [including rotational motion of N_2]

We find $\frac{dT}{dh} = -9.8$ deg/km

(c) This prediction overestimates the temperature gradient in the lower atmosphere by about 50%. (The true gradient is closer to $-6.5°$/km.) It is primarily the condensation of water vapor that releases heat and slows down the falloff in temperature.

C+C Prob. 18

(3) This problem is actually solved on pp. 58–59 of the text. I will redo the solution in a more physically instructive way.

The star's radius is $R = R_0 \xi_1$ where $R_0 \equiv \left[\frac{(n+1)K}{4\pi G \rho_0^{1-1/n}}\right]^{1/2}$,

and where ξ_1 is the first zero of the Lane-Emden equation. (See eq. (5.61) in text.) From eq. (5.63), we also find the mass:

$$M = -4\pi \left(\xi^2 \frac{d\theta}{d\xi}\right)_1 \rho_0 R_0^3$$

[The integral in eq. (5.63) can be done using Lane-Emden equation.]

Dimensionally, the central pressure $P_0 \left(= K\rho_0^{1+1/n}\right)$ must scale as $\frac{GM^2}{R^4}$. In fact, we find from above:

$$\frac{GM^2}{R^4} = \frac{G\,16\pi^2 \left(\xi^2 \frac{d\theta}{d\xi}\right)_1^2 \rho_0^2 R_0^6}{\xi_1^4 R_0^4} = 16\pi^2 \left(\frac{d\theta}{d\xi}\right)_1^2 G \rho_0^2 R_0^2$$

substituting for R_0, $\frac{GM^2}{R^4} = 4\pi (n+1) \left(\frac{d\theta}{d\xi}\right)_1^2 K \rho_0^{1+1/n}$

$$= 4\pi (n+1) \left(\frac{d\theta}{d\xi}\right)_1^2 P_0$$

We also know that ρ_0 must scale as M/R^3. We find

$$\frac{M}{R^3} = \frac{-4\pi \left(\xi^2 \frac{d\theta}{d\xi}\right)_1 \rho_0 R_0^3}{\xi_1^3 R_0^3} = -\frac{4\pi}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1 \rho_0$$

Thus $$\boxed{\frac{GM}{R} = -(n+1)\left(\xi \frac{d\theta}{d\xi}\right)_1 \frac{P_0}{\rho_0} = -(n+1)\left(\xi \frac{d\theta}{d\xi}\right)_1 \frac{RT_0}{\mu}}$$

If T_0 is fixed for all stars, then $M \propto R$.

C&C Prob. 22

(4) (a) $P_0 = A\rho_0^{1+1/n}$ $\qquad \rho_0 = (kx)^n$

Since x points downward, we have $\dfrac{1}{\rho_0}\dfrac{dP_0}{dx} = g$

Now $\dfrac{dP_0}{dx} = \left(1+\frac{1}{n}\right)A\rho_0^{1/n}\dfrac{d\rho_0}{dx}$ and $\dfrac{d\rho_0}{dx} = \dfrac{n\rho_0}{x}$

Thus $g = \dfrac{\left(1+\frac{1}{n}\right)A\rho_0^{1/n}\,n\rho_0}{x\rho_0} = \dfrac{(n+1)A\rho_0^{1/n}}{x}$

But $\rho_0^{1/n} = kx$, so $\boxed{g = Ak(1+n)}$

NB: Solving for the coefficient k, and noting that $n = 1/(\gamma-1)$, we have the same density profile as in Problem 1.

(b) The relation $P = A\rho^{1+1/n}$ remains valid during an adiabatic perturbation, thus

$$\Delta P = \left(1+\frac{1}{n}\right)A\rho_0^{1/n}\,\Delta\rho = \frac{A(n+1)}{n}\rho_0^{1/n}\,\Delta\rho$$

Again using $\rho_0^{1/n} = kx$, $\Delta P = \dfrac{Ak(1+n)x}{n}\Delta\rho \longrightarrow \boxed{\Delta P = \frac{gx}{n}\Delta\rho}$

(c) Continuity equation is now $\dfrac{\partial\rho}{\partial t} - \dfrac{\partial}{\partial x}(\rho u) = 0$

$$\frac{\partial\rho}{\partial t} - u\frac{\partial\rho}{\partial x} - \rho\frac{\partial u}{\partial x} = 0$$

Perturbing this from equilibrium,

$$\boxed{\frac{\partial\Delta\rho}{\partial t} - u\frac{\partial\rho_0}{\partial x} - \rho_0\frac{\partial u}{\partial x} = 0}$$

NB: This equation is correct for Eulerian perturbations, $\delta\rho$ and δu. It is true that $\delta u = \Delta u$, the Lagrangian velocity perturbation. However, $\delta\rho \neq \Delta\rho$, as the book emphasizes in Chapter 6. In fact, we have $\partial(\delta\rho)/\partial t = \partial\Delta\rho/\partial t + u\,\partial\rho_0/\partial x$. The above result is therefore wrong. In light of this error, I simply ignored the Eulerian/Lagrangian distinction in the remainder of this solution — and in the grading.

(d) Euler's equation is $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = +\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} - g$

The perturbed version is

$$\frac{\partial u}{\partial t} = \frac{1}{\rho_0}\frac{\partial \Delta p}{\partial x} - \frac{1}{\rho_0^2}\Delta\rho\,\frac{\partial p_0}{\partial x} \qquad \left[\text{since } \Delta\left(\frac{1}{\rho}\right) = -\frac{1}{\rho_0}\Delta\rho\right]$$

But $\Delta p = \dfrac{gx}{n}\Delta\rho \rightarrow \dfrac{\partial \Delta p}{\partial x} = \dfrac{g}{n}\dfrac{\partial}{\partial x}(x\Delta\rho)$

$$\boxed{\frac{\partial u}{\partial t} = \frac{g}{n\rho_0}\frac{\partial}{\partial x}(x\Delta\rho) - \frac{\Delta\rho}{\rho_0}g} \qquad \text{since } \frac{1}{\rho_0}\frac{\partial p_0}{\partial x} = g$$

(e) Taking $\partial/\partial t$ of the above equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{g}{n\rho_0}\frac{\partial \Delta\rho}{\partial t} + \frac{gx}{n\rho_0}\frac{\partial^2\Delta\rho}{\partial x\,\partial t} - \frac{g}{\rho_0}\frac{\partial\Delta\rho}{\partial t}$$

To evaluate $\partial^2\Delta\rho/\partial x\,\partial t$, we take $\frac{\partial}{\partial x}$ of the perturbed continuity eqtn:

$$\frac{\partial^2\Delta\rho}{\partial x\,\partial t} = \frac{\partial u}{\partial x}\frac{\partial \rho_0}{\partial x} + u\frac{\partial^2\rho_0}{\partial x^2} + \frac{\partial\rho_0}{\partial x}\frac{\partial u}{\partial x} + \rho_0\frac{\partial^2 u}{\partial x^2}$$

Plugging in,

$$\frac{\partial^2 u}{\partial t^2} = \frac{g}{\rho_0}\left(\frac{1}{n}-1\right)\frac{\partial\Delta\rho}{\partial t} + \frac{gx}{n\rho_0}\left[\frac{\partial^2\rho_0}{\partial x^2}u + 2\frac{\partial\rho_0}{\partial x}\frac{\partial u}{\partial x} + \rho_0\frac{\partial^2 u}{\partial x^2}\right]$$

We also evaluate $\partial\Delta\rho/\partial t$ from the perturbed continuity eqtn:

$$\frac{\partial^2 u}{\partial t^2} = \frac{g}{\rho_0}\left(\frac{1}{n}-1\right)\left(u\frac{\partial\rho_0}{\partial x} + \rho_0\frac{\partial u}{\partial x}\right) + \frac{gx}{n\rho_0}\left[\frac{\partial^2\rho_0}{\partial x^2}u + 2\frac{\partial\rho_0}{\partial x}\frac{\partial u}{\partial x} + \rho_0\frac{\partial^2 u}{\partial x^2}\right]$$

$$\boxed{\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= u\left[\frac{g}{\rho_0}\left(\frac{1}{n}-1\right)\frac{\partial\rho_0}{\partial x} + \frac{gx}{\rho_0 n}\frac{\partial^2\rho_0}{\partial x^2}\right] \\ &+ \frac{\partial u}{\partial x}\left[\frac{g}{\rho_0}\left(\frac{1}{n}-1\right)\rho_0 + \frac{2gx}{\rho_0 n}\frac{\partial\rho_0}{\partial x}\right] \\ &+ \frac{\partial^2 u}{\partial x^2}\left(\frac{gx}{n}\right)\end{aligned}}$$

NB: $\rho_0 = k^n x^n$

$$\frac{d\rho_0}{dx} = \frac{n\rho_0}{x}$$

$$\frac{d^2\rho_0}{dx^2} = \frac{n(n-1)\rho_0}{x^2}$$

For $n = 1/2$, we indeed find $\boxed{\dfrac{\partial^2 u}{\partial t^2} = 3g\dfrac{\partial u}{\partial x} + 2gx\dfrac{\partial^2 u}{\partial x^2}}$

(f) If $u = u_1 e^{i\omega t}$, then $\frac{\partial^2 u}{\partial t^2} = -\omega^2 u$, and the above eqn is

$$-\omega^2 u_1 = 2gx \frac{\partial^2 u_1}{\partial x^2} + 3g \frac{\partial u_1}{\partial x}$$

Let $\tau \equiv \left(\frac{2x}{g}\right)^{1/2} \rightarrow x = g\tau^2/2 \longrightarrow \frac{d\tau}{dx} = \frac{1}{g\tau}$

Thus, $\frac{du_1}{dx} = \frac{d\tau}{dx}\frac{du_1}{d\tau} = \frac{1}{g\tau}\frac{du_1}{d\tau}$ $\quad \frac{d^2u_1}{dx^2} = \frac{1}{g\tau}\frac{d}{d\tau}\left[\frac{1}{g\tau}\frac{du_1}{d\tau}\right]$

$$= \frac{1}{g^2\tau^2}\left[-\frac{1}{\tau}\frac{du_1}{d\tau} + \frac{d^2u_1}{d\tau^2}\right]$$

Plugging in,

$$-\omega^2 u_1 = \frac{d^2u_1}{d\tau^2} + \frac{2}{\tau}\frac{du_1}{d\tau}$$

Now $\frac{d}{d\tau}(\tau u_1) = u_1 + \tau\frac{du_1}{d\tau}$

$$\frac{d^2}{d\tau^2}(\tau u_1) = \tau\frac{d^2u_1}{d\tau^2} + 2\frac{du_1}{d\tau}$$

thus, $\frac{1}{\tau}\frac{d^2}{d\tau^2}(\tau u_1) = \frac{d^2u_1}{d\tau^2} + \frac{2}{\tau}\frac{du_1}{d\tau}$

$$\Downarrow$$

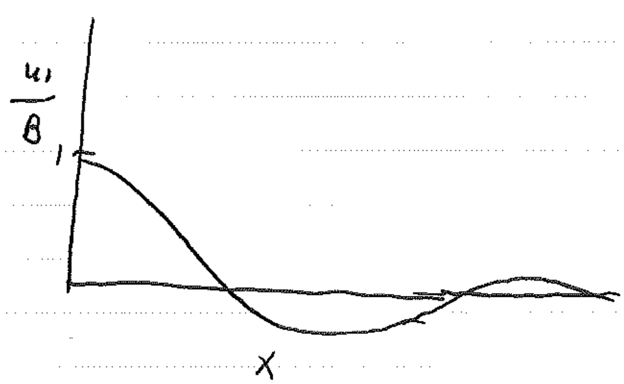
$$\boxed{-\omega^2 u_1 = \frac{1}{\tau}\frac{d^2}{d\tau^2}(\tau u_1)}$$

(g) We have $\frac{d^2(\tau u_1)}{d\tau^2} + \omega^2(\tau u_1) = 0$ The general solution is

$$\tau u_1 = A\cos\omega\tau + B\sin\omega\tau$$

Since $\tau \to 0$ as $x \to 0$, we need $\underline{A = 0}$ to prevent divergence of u_1.

So $$\boxed{u_1 = B\sqrt{\frac{g}{2x}}\sin\omega\sqrt{\frac{2x}{g}}}$$



$u_1(x)$ is a decaying oscillation in distance x. A wave launched at the top of the atmosphere cannot readily perturb material far below, where the density climbs higher