AY 202
Stahler

Assignment 3 Solutions

① For incompressible $(\vec{\nabla}\cdot\vec{u}=0)$, steady-state $\left(\frac{\partial}{\partial t}=0\right)$ flow, the Navier-Stokes equation is

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{u}$$

We may use Appendix A.4 of C&C to look up the components of $(\vec{u}\cdot\vec{\nabla})\vec{u}$ in cylindrical coordinates. We also need the given expression for the $\nabla^2\vec{u}$ components. Finally, we use the fact that $u_R = u_z = 0$, and that the flow is axisymmetric $(\partial/\partial\varphi = 0)$

For the R-component, $-\dfrac{u_\varphi^2}{R} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial R}$

while the φ-component is $0 = \nu\left[\nabla^2\vec{u}\right]_\varphi = \dfrac{\nu}{R}\dfrac{\partial}{\partial R}\left(R\dfrac{\partial u_\varphi}{\partial R}\right) - \dfrac{\nu u_\varphi}{R^2}$

We will solve the 2nd equation for $u_\varphi(R)$. If desired, we can plug this expression into the first equation to find $p(R)$ — although the problem does not ask for the pressure distribution.

The second equation is $\boxed{0 = \dfrac{\partial^2 u_\varphi}{\partial R^2} + \dfrac{1}{R}\dfrac{\partial u_\varphi}{\partial R} - \dfrac{u_\varphi}{R^2}}$

Try a power law: $u_\varphi = CR^n$ where C and n are constants. We find

$$n(n-1)R^{n-1} + nR^{n-1} - R^{n-1} = 0 \quad \text{(after cancelling } C)$$

$$n(n-1) + n - 1 = 0 \rightarrow n^2 = 1 \rightarrow \boxed{n = \pm 1}$$

Thus $u_\varphi = AR + B/R$ where A and B are constants. These constants are found by using the no-slip conditions at the two cylindrical walls:

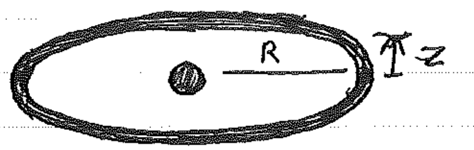
$$u_\varphi(R_1) = R_1\Omega_1 \qquad u_\varphi(R_2) = R_2\Omega_2$$

We find

$$A = \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2} \qquad B = \frac{R_1^2R_2^2(\Omega_1 - \Omega_2)}{R_2^2 - R_1^2} \quad \text{so that}$$

$$\boxed{u_\varphi = \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2}R + \frac{R_1^2R_2^2(\Omega_1 - \Omega_2)}{R(R_2^2 - R_1^2)}}$$

② C&C, problem 14



In the plane: $\Omega^2 R = \frac{GM}{R^2}$

Normally: $g_\perp = \frac{GMz}{R^3} = \frac{-a_T^2}{\rho}\frac{d\rho}{dz}$

(a) $$\frac{1}{\rho}\frac{d\rho}{dz} = \frac{d\ln\rho}{dz} = \frac{-GMz}{a_T^2 R^3} = \frac{-\Omega^2 z}{a_T^2}$$

(The expression for $g_\perp$ only holds for $z \ll R$.)
Integrating to find $\rho(z)$:

$$\ln\rho - \ln\rho_0 = \frac{-\Omega^2 z^2}{2a_T^2}$$ where ρ_0 is midplane density

$$\rho = \rho_0 e^{-\frac{\Omega^2 z^2}{2a_T^2}} = \rho_0 e^{-z^2/H^2}$$

where the e-folding height $\boxed{H \equiv \frac{\sqrt{2}\, a_T}{\Omega} = \sqrt{\frac{2\mathcal{R}T}{\mu\Omega^2}}}$

since $a_T = \sqrt{\frac{\mathcal{R}T}{\mu}}$

(b) for $H \ll R$, $\frac{2\mathcal{R}T}{\mu\Omega^2} \ll R^2 \longrightarrow \frac{2\mathcal{R}T}{\mu} \ll R^2\Omega^2 = V_\varphi^2$

where V_φ is the Earth's orbital velocity (30 km s^{-1})
Using $\mu = 1$ (atomic hydrogen),

$$\boxed{T \ll \frac{\mu V_\varphi^2}{2\mathcal{R}} = 5\times10^4\ \mathrm{K}}$$

③ C&C, problem 32

(a) $u\dfrac{du}{dx} = -\dfrac{1}{\rho}\dfrac{dp}{dx}$ for steady-state flow

Now $p = k\rho^2 \longrightarrow -\dfrac{1}{\rho}\dfrac{dp}{dx} = -2k\dfrac{d\rho}{dx} = -2\dfrac{p_0}{\rho_0^2}\dfrac{d\rho}{dx}$

We have $\dfrac{d}{dx}\left(\tfrac{1}{2}u^2\right) = -\dfrac{2p_0}{\rho_0^2}\dfrac{d\rho}{dx}$ Integrating,

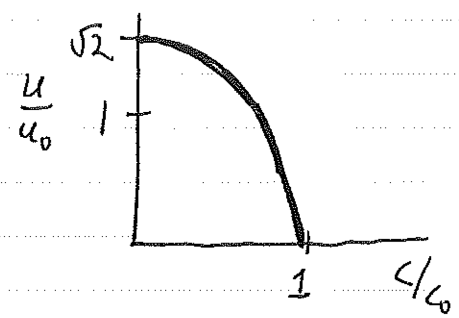
$$\tfrac{1}{2}u^2 - \tfrac{1}{2}u_0^2 = -\frac{2p_0}{\rho_0^2}(\rho-\rho_0)$$

The sound speed c is found from $c^2 = \dfrac{2p}{\rho} = c_0^2\dfrac{p}{p_0}\dfrac{\rho_0}{\rho} = c_0^2\dfrac{\rho^2}{\rho_0^2}\dfrac{\rho_0}{\rho}$

where $c_0^2 \equiv 2p_0/\rho_0$ $= c_0^2\dfrac{\rho}{\rho_0}$

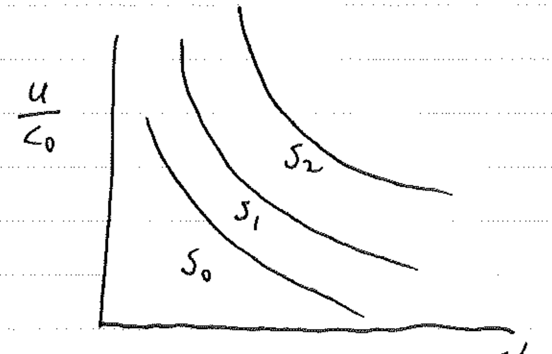
Thus $\tfrac{1}{2}u^2 - \tfrac{1}{2}u_0^2 = -c_0^2\left(\dfrac{\rho}{\rho_0}-1\right) = -c_0^2\left(\dfrac{c^2}{c_0^2}-1\right) = c_0^2 - c^2$

Since $u_0^2 \ll c_0^2$, we have approximately $\boxed{\tfrac{1}{2}u^2 + c^2 = c_0^2}$



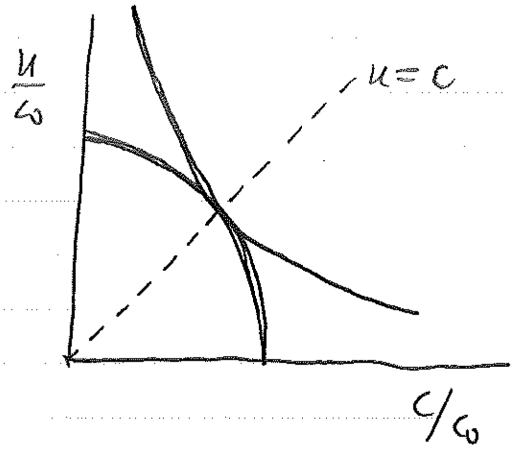
(b) $\rho u = \dfrac{\dot m}{S}$ But $\rho = \dfrac{c^2}{c_0^2}\rho_0$

$$\boxed{c^2 u = \frac{\dot m\, c_0^2}{\rho_0 S}}$$



$(S_0 > S_1 > S_2)$

(c) The problem assumes (but unfortunately does not state) that a sonic transition is made. The only way this can be done smoothly is for the $S = S_{min}$ curve to touch the original u-c curve, as shown.



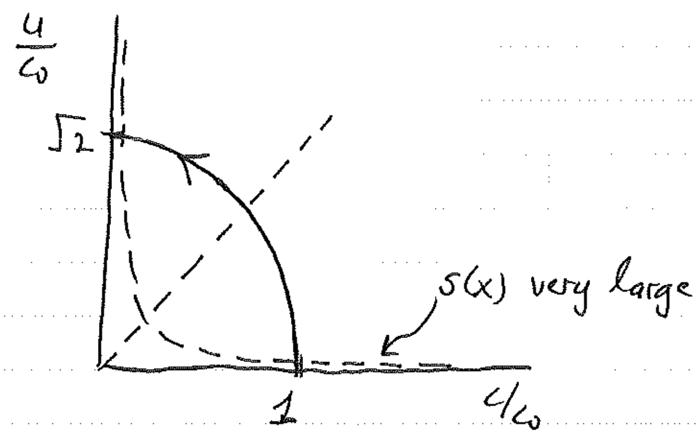
Just at the throat, $u = c$, so that

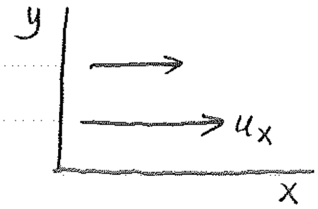
$$\frac{1}{2}u^2 + c^2 = \frac{1}{2}u^2 + u^2 = \frac{3}{2}u^2 = c_0^2 \longrightarrow \boxed{u = \sqrt{\frac{2}{3}}\, c_0}$$

(d) As $S(x)$ broadens, the representative point slides along the original u-c curve. The point asymptotically approaches

$$u \to \sqrt{2}\, c_0$$

as $S(x) \to \infty$.





(4) C & C Problem 45

Since there is no pressure gradient, the two components of Navier–Stokes are:

$$(1)\quad \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \left\{ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{1}{3}\frac{\partial}{\partial x}\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right]\right\}$$

$$(2)\quad \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = \nu \left\{ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{1}{3}\frac{\partial}{\partial y}\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right]\right\}$$

Eqtn (2) is identically satisfied if $u_y = 0$ at all times <u>and</u> if $\partial u_x/\partial x = 0$ at all times. Assuming both conditions hold, eqtn (1) becomes

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial x^2}$$

which is the diffusion equation. Let $u_x = f(t)\, g(y)$

Then $\frac{1}{\nu f}\frac{df}{dt} = \frac{1}{g}\frac{d^2 g}{dy^2} = k^2$ where k can be positive or negative constant

Solution: $f = e^{-k^2 \nu t}$, $g = e^{iky}$

So, more generally, we may write

$$u_x(y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{iky}\, e^{-k^2\nu t}\, dk$$

To determine $A(k)$, consider the initial condition:

$$u_x(y,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{iky}\, dk$$

Thus, $A(k)$ is the Fourier transform:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_x(y',0)\, e^{-iky'}\, dy'$$

Plugging this expression back into the integral for $u_x(y,t)$:

$$u_x(y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy'\, u_x(y',0) \int_{-\infty}^{\infty} dk\, e^{-[k^2\nu t - ik(y-y')]}$$

We complete the square in the exponential:

$$k^2 \nu t - ik(y-y') = \left[k\sqrt{\nu t} - \frac{i(y-y')}{\sqrt{4\nu t}}\right]^2 + \frac{(y-y')^2}{4\nu t}$$

Thus,

$$u_x(y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy'\, u_x(y',0)\, e^{-\frac{(y-y')^2}{4\nu t}} \int_{-\infty}^{\infty} dk\, e^{-\left[k\sqrt{\nu t} - \frac{i(y-y')}{\sqrt{4\nu t}}\right]^2}$$

In the last integral, y, y', and t are all constants. Let $x \equiv k\sqrt{\nu t} - \frac{i(y-y')}{\sqrt{4\nu t}}$
Then $dk = \frac{dx}{\sqrt{\nu t}}$, and

$$u_x(y,t) = \frac{1}{2\pi\sqrt{\nu t}}\int_{-\infty}^{\infty} dy'\, u_x(y',0)\, e^{-\frac{(y-y')^2}{4\nu t}} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-x^2}}_{\sqrt{\pi}}$$

$$\boxed{u_x(y,t) = \frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{\infty} dy'\, u_x(y',0)\, e^{\frac{-(y-y')^2}{4\nu t}}}$$

which is the book's solution