AY 202
Stahler

Assignment 4 Solutions

① (a) The R-component of Navier-Stokes is $\frac{1}{\rho}\frac{\partial p}{\partial R} = \frac{u^2}{R}$

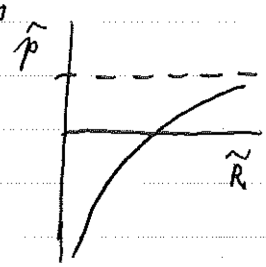
Initially, $u = \frac{k}{2\pi R}$

Thus $\frac{\partial p}{\partial R} = \frac{k^2 \rho}{4\pi^2 R^3}$ Since p_0 is the pressure at large R,

$$p_0 - p = \int_R^\infty \frac{\partial p}{\partial R} dR = \frac{k^2}{4\pi^2 \rho}\int_R^\infty R^{-3} dR = \frac{k^2}{8\pi^2 \rho R^2}$$

Or $\boxed{p = p_0 - \frac{k^2 \rho}{8\pi^2 R^2}}$ Which we can write as

$$\tilde{p} = 1 - \frac{1}{\tilde{R}^2}$$



Here $\tilde{p} \equiv \frac{p}{p_0}$, $\tilde{R} = \frac{R}{R_0}$, $R_0^2 \equiv \frac{\rho k^2}{8\pi^2 p_0}$

(b) $u(R,t) = \frac{\Gamma}{2\pi R} = \frac{k}{2\pi R}\left[1 - \exp\left(-\frac{R^2}{4\nu t}\right)\right]$

$$\frac{\rho u^2}{R} = \frac{\rho k^2}{4\pi^2 R^3}\left[1 - \exp\left(-\frac{R^2}{4\nu t}\right)\right]^2 = \frac{dp}{dR} = 2R\frac{dp}{dR^2}$$

$$2R^4 \frac{dp}{dR^2} = \frac{\rho k^2}{4\pi^2}\left[1 - \exp\left(-\frac{R^2}{4\nu t}\right)\right]^2$$ Let $y \equiv \frac{R^2}{4\nu t}$. Then

$$\frac{dp}{dy} = \frac{\rho k^2}{32\pi^2 \nu t}\frac{(1-e^{-y})^2}{y^2}$$ Integrating as before,

$$p_0 - p = \frac{\rho k^2}{32\pi^2 \nu t}\int_y^\infty \left(\frac{1}{y^2} - \frac{2e^{-y}}{y^2} + \frac{e^{-2y}}{y^2}\right) dy$$ Doing the integrals:

$$\int_y^\infty \frac{1}{y^2} dy = \frac{1}{y} \qquad \int_y^\infty e^{-y}\frac{dy}{y^2} = \left| -\frac{e^{-y}}{y}\right|_y^\infty - \int_y^\infty \left(-\frac{1}{y}\right)\left(-e^{-y}\right) dy$$

$$= \frac{e^{-y}}{y} - \int_y^\infty \frac{e^{-y}}{y} dy = \frac{e^{-y}}{y} - E_1(y)$$

$$\int_y^\infty e^{-2y}\frac{dy}{y^2} = \left| -\frac{e^{-2y}}{y}\right|_y^\infty - \int_y^\infty \left(-\frac{1}{y}\right)\left(-2e^{-2y}\right) dy = \frac{e^{-2y}}{y} - 2\int_y^\infty \frac{e^{-2y}}{y} dy$$

$$= \frac{e^{-2y}}{y} - 2\int_{2y}^\infty \frac{e^{-2y}}{2y} d(2y) = \frac{e^{-2y}}{y} - 2E_1(2y)$$

Putting this all together: $$\boxed{p = p_0 - \frac{\rho k^2}{32\pi^2 \nu t}\left[\frac{1}{y}\left(1-e^{-y}\right)^2 + 2E_1(y) - 2E_1(2y)\right]}$$

(c) We already defined

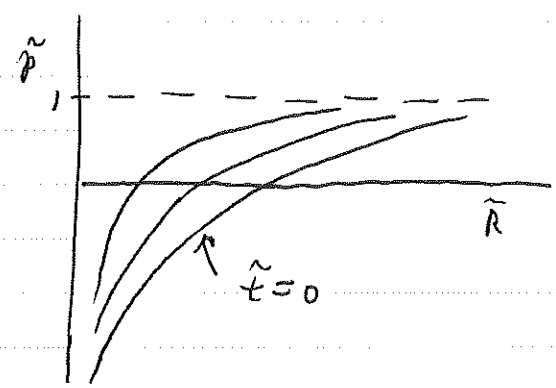
$$R_0^2 \equiv \frac{\rho k^2}{8\pi^2 p_0} \qquad \text{Now } y \equiv \frac{R^2}{4\nu t} = \frac{R_0^2}{4\nu t}\tilde{R}^2 = \frac{\tilde{R}^2}{\tilde{t}}$$

$$\text{where } \tilde{t} \equiv \frac{t}{t_0} \text{ and } t_0 \equiv \frac{R_0^2}{4\nu}$$

Finally $$\frac{\rho k^2}{32\pi^2 \nu t} = \frac{\rho k^2}{8\pi^2 p_0}\frac{p_0}{4\nu t} = \frac{R_0^2}{4\nu t}p_0 = \frac{p_0}{\tilde{t}}$$

Thus, $p(R,t)$ can be written: $$\boxed{\tilde{p} = 1 - \frac{1}{\tilde{t}}\left[\frac{1}{y}\left(1-e^{-y}\right)^2 + 2E_1(y) - 2E_1(2y)\right]}$$

where $y \equiv \tilde{R}^2/\tilde{t}$



where higher curves are at later $\tilde{t}$

② (a) The 1st law, for an ideal gas, is

$$T\,ds = c_v\,dT + p\,d(1/\rho)$$

Since $p = \rho \mathcal{R} T/\mu$, we also have $\frac{\mathcal{R}}{\mu}\,dT = p\,d\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\,dp$

Thus,

$$ds = \frac{c_v\,p\,d(1/\rho)}{\mathcal{R}T/\mu} + \frac{c_v\,dp}{\rho \mathcal{R}T/\mu} + \frac{p}{T}\,d\left(\frac{1}{\rho}\right)$$

$$ds = c_v\,\rho\,d(1/\rho) + c_v\,dp/p - \frac{p}{\rho^2 T}\,d\rho$$

$$ds = -c_v\frac{d\rho}{\rho} + c_v\frac{dp}{p} - \frac{\mathcal{R}}{\mu}\frac{d\rho}{\rho}$$

But we also know that $c_p = \gamma c_v = c_v + \frac{\mathcal{R}}{\mu}$, so that

$$ds = -\gamma c_v\frac{d\rho}{\rho} + c_v\frac{dp}{p}$$ Integrating,

$$\boxed{s = c_v \ln\left(p\,\rho^{-\gamma}\right) + s_0}$$ where s_0 is a constant

(b) Across the shock, $\Delta s = c_v \ln\left[\left(\frac{p_2}{p_1}\right)\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma}\right]$

$$e^{\Delta s/c_v} = \left(\frac{p_2}{p_1}\right)\left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

We need to express ρ_2/ρ_1 in terms of p_2/p_1. From class,

$$\left(1-\frac{p_2}{p_1}\right)\left(1+\frac{\rho_2}{\rho_1}\right) = \frac{2\gamma}{\gamma-1}\left(\frac{\rho_2}{\rho_1} - \frac{p_2}{p_1}\right)$$ so that

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma-1) + (\gamma+1)(p_2/p_1)}{(\gamma-1)(p_2/p_1) + (\gamma+1)}$$

Thus $$\boxed{e^{\Delta s/c_v} = \left(\frac{p_2}{p_1}\right)\left[\frac{(\gamma-1)(p_2/p_1) + (\gamma+1)}{(\gamma+1)(p_2/p_1) + (\gamma-1)}\right]^{\gamma}}$$

(c) When $x \equiv p_2/p_1 \to 1$, the foregoing equation shows that $\exp(\Delta s/c_v) \to 1$, so that $\Delta s \to 0$. We want to show that $\Delta s < 0$ for $x < 1$. We do this by showing that $ds/dx > 0$ for all x. (NB: $ds/dx = d\Delta s/dx$).

From the last equation, $\frac{1}{c_v}\frac{ds}{dx} = \frac{1}{x} + \frac{\gamma(\gamma-1)}{(\gamma-1)x + (\gamma+1)} - \frac{\gamma(\gamma+1)}{(\gamma+1)x + (\gamma-1)}$

Let $\frac{1}{c_v}\frac{ds}{dx} = \frac{N}{D}$. Here, $D = x[(\gamma-1)x + (\gamma+1)][(\gamma+1)x + (\gamma-1)]$

Since $x > 0$ (and $\gamma > 1$), it can be seen that $D > 0$.

What about N?

$$N = [(\gamma-1)x + (\gamma+1)][(\gamma+1)x + (\gamma-1)] + \gamma(\gamma-1)x[(\gamma+1)x + (\gamma-1)] - \gamma(\gamma+1)x[(\gamma-1)x + (\gamma+1)]$$

$$N = (\gamma^2-1)x^2 + 2(\gamma^2+1)x + (\gamma^2-1) + \gamma x[(\gamma-1)^2 - (\gamma+1)^2]$$

$$N = \gamma^2x^2 - x^2 - 2\gamma^2 x + 2x + \gamma^2 - 1$$

$$N = (\gamma^2-1)(x-1)^2 > 0$$

Thus, $\frac{1}{c_v}\frac{ds}{dx} > 0$. So for $x < 1$, $\Delta s < 0$, which violates the Second Law of thermodynamics.

(3) C & C Problem 27 (a) Consider the quantity $\frac{(\rho u^2)_2}{p_2} \frac{p_1}{(\rho u^2)_1} \equiv R$

Since $M_1^2 = \frac{u_1^2}{a_1^2} = \frac{\rho_1 u_1^2}{\gamma p_1}$, we have $R = \frac{1}{\gamma M_1^2} \frac{(\rho u^2)_2}{p_2}$

But also, $R = \frac{\rho_2}{\rho_1}\left(\frac{u_2}{u_1}\right)^2 \frac{p_1}{p_2} = \frac{u_2}{u_1}\frac{p_1}{p_2}$, since $(\rho u)_1 = (\rho u)_2$

from lecture, $\frac{u_2}{u_1} = \frac{2 + (\gamma-1) M_1^2}{(\gamma+1) M_1^2}$ $\qquad \frac{p_1}{p_2} = \frac{\gamma+1}{2\gamma(M_1^2-1) + (\gamma+1)}$

Thus $R = \frac{1}{\gamma M_1^2}\frac{(\rho u^2)_2}{p_2} = \frac{2 + (\gamma-1) M_1^2}{M_1^2\left[2\gamma(M_1^2-1) + (\gamma+1)\right]}$

and we have: $\boxed{\frac{(\rho u^2)_2}{p_2} = \frac{\gamma\left[2 + (\gamma-1) M_1^2\right]}{2\gamma(M_1^2-1) + (\gamma+1)}}$

Consider first a <u>weak</u> shock. As $M_1 \to 1$ from above,

$$\frac{(\rho u^2)_2}{p_2} \to \frac{\gamma\left[2 + (\gamma-1)\right]}{\gamma+1} = \gamma > 1$$

Next consider (as does the problem) a <u>strong</u> shock. For $M_1^2 >> 1$,

$$\boxed{\frac{(\rho u^2)_2}{p_2} \to \frac{\gamma(\gamma-1)M_1^2}{2\gamma M_1^2} = \frac{\gamma-1}{2}} \left(= \tfrac{1}{3} \text{ for } \gamma = 5/3\right)$$

(b) In the postshock cooling region, $p + \rho u^2 = p_2 + (\rho u^2)_2$

Thus $\frac{p}{p_2} = 1 + \frac{(\rho u^2)_2}{p_2} - \frac{\rho u^2}{p_2}$

The ratio p/p_2 starts at <u>1</u> and then climbs as ρu^2 falls below p_2.
But it can never exceed $1 + \frac{(\rho u^2)_2}{p_2} = 1.33$ for $\gamma = 5/3$ and a strong shock

(c) $c_p \frac{dT}{dt} = -\dot{q} = -kT^2 \longrightarrow \frac{1}{T^2}\frac{dT}{dt} = \frac{-k}{c_p}$ Integrating

$-\frac{1}{T} + \frac{1}{T_2} = \frac{-k}{c_p} t$ $\qquad \boxed{T = T_2\left(1 + \frac{kT_2}{c_p} t\right)^{-1}}$

(d) Behind the shock, $\rho u = \rho_2 u_2 =$ constant

Since p is assumed constant, $\rho \propto T^{-1}$

Thus $u \propto T$, and $\boxed{u = \dfrac{u_2 T}{T_2}}$ as claimed

(e) $$u = \frac{dx}{dt} = \frac{u_2 T(t)}{T_2} = u_2\left(1 + \frac{kT_2}{c_p}t\right)^{-1}$$

$$x = u_2 \int_0^t \frac{dt}{1 + \frac{kT_2}{c_p}t} = \frac{c_p u_2}{kT_2}\int_0^{\frac{kT_2}{c_p}t} \frac{dx}{1+x}$$

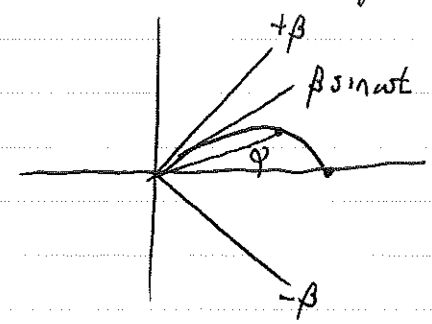
$$= \frac{c_p u_2}{kT_2} \ln\left[1 + \frac{kT_2}{c_p}t\right]$$

Thus, $\exp\left[\dfrac{kT_2 x}{c_p u_2}\right] = 1 + \dfrac{kT_2}{c_p}t$, and

$$\boxed{T = T_2\left(1 + \frac{kT_2}{c_p}t\right)^{-1} = T_2 \exp\left[-\frac{kT_2}{c_p u_2}x\right]}$$

④ C&C, Problem 33

(a) Since $V_R >> C_s$, we can consider each fluid element following a ballistic trajectory. At any time t, the jet has rotated to the angle $\beta \sin \omega t$, where $\omega = 2\pi/P$. Consider a fluid element located at $\varphi < \beta \sin \omega t$. For how long a time, t', has that element been traveling? The element was emitted at time $t - t'$. At that time, the jet had rotated to $\varphi = \beta \sin \omega (t - t')$. Thus

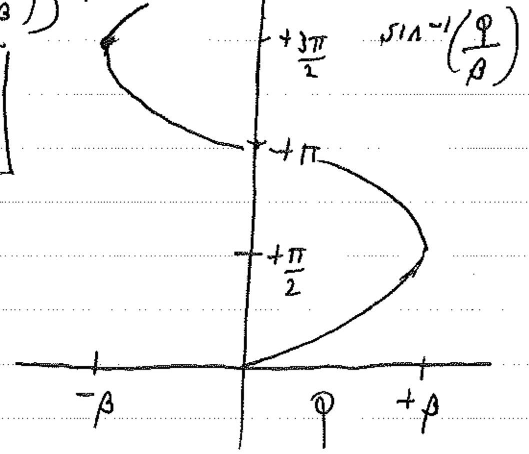


$$t' = t - \frac{1}{\omega} \sin^{-1}\left(\frac{\varphi}{\beta}\right)$$

The element has traveled a radial distance $r = V_R t'$. Thus, the locus of all fluid elements is given by

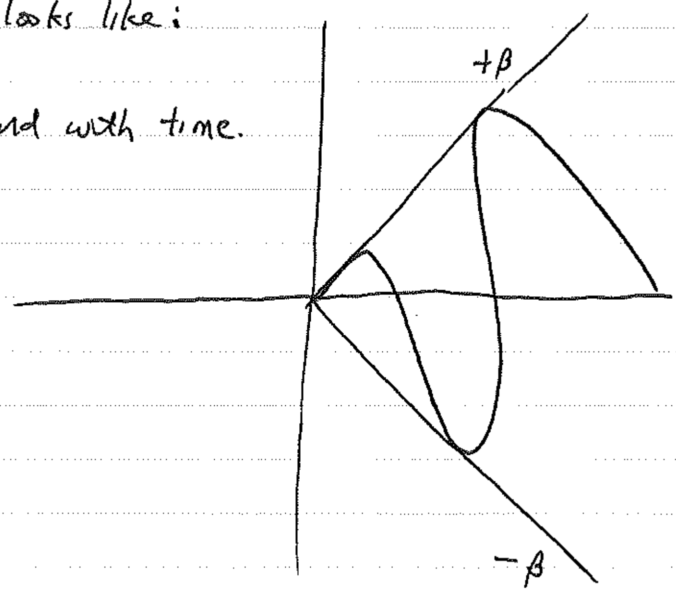
$$r = V_R \left[t - \frac{1}{\omega} \sin^{-1}\left(\frac{\varphi}{\beta}\right) \right]$$, which may be recast as

$$\boxed{\frac{\omega r}{V_R} = \omega t - \sin^{-1}\left(\frac{\varphi}{\beta}\right)}$$



For every ωt, we can find the range of φ such that $\omega r / V_R$ goes from 0 to ωt. The φ-values can be as small as $-\beta$ and as large as $+\beta$.

The curve $r(\varphi)$ looks like:



and expands outward with time.

(b) $V_R(t) = V_0 + V_1 \sin \omega t$, $\omega \equiv 2\pi/p$

Pressure can no longer be ignored when two fluid elements overtake one another. At $t=0$, an element leaves the origin with speed V_0. At $t=t_1$, it leaves with speed $V_0 + V_1 \sin \omega t_1$. The second element overtakes the first at time t_2, where

$$V_0 t_2 = (V_0 + V_1 \sin \omega t_1)(t_2 - t_1) \qquad \text{so that}$$

$$t_2 = \frac{t_1 [V_0 + V_1 \sin \omega t_1]}{V_1 \sin \omega t_1}$$

The interaction distance is

$$\boxed{R = V_0 t_2 = \frac{V_0^2}{\omega V_1} \frac{(\omega t_1)\left(1 + \frac{V_1}{V_0} \sin \omega t_1\right)}{\sin \omega t_1}}$$

For small t_1, $\boxed{R \simeq \frac{V_0^2}{\omega V_1}}$

This is the minimum distance. For example, if $\omega t_1 = \pi/2$,

$$R = \frac{V_0^2}{\omega V_1} \frac{\pi}{2}\left(1 + \frac{V_1}{V_0}\right) > \frac{V_0^2}{\omega V_1}$$