AY 202 Assignment 5

due: Tuesday, April 5

Problem 1: A rotating fluid tends to develop coherent substructures along the direction of the rotation axis. Such *Taylor columns* are easily produced in the laboratory. Let us explore the basis of the phenomenon.

(a) Consider motions within an incompressible, inviscid, rotating fluid. View the fluid in a reference frame rotating with the angular velocity $\boldsymbol{\Omega}$. If $\boldsymbol{u}$ is the fluid velocity in this frame, write out the conditions of mass and momentum conservation. For the latter, let ρ, p, and Φ be the density, pressure, and gravitational potential, respectively.

(b) Define the reduced pressure p' as

$$p' \equiv p + \rho\,\Phi - \frac{1}{2}\,\rho\,|\boldsymbol{\Omega}\times\boldsymbol{r}|^2 \ .$$

Using p' instead of p, derive a simplified version of the momentum equation.

(c) Now consider steady-state motion, slow enough that you can neglect the convective portion of the acceleration $d\boldsymbol{u}/dt$. What is the momentum equation now?

(d) Take the curl of this last equation and utilize mass conservation to derive an expression for the change in $\boldsymbol{u}$ in the direction of $\boldsymbol{\Omega}$. This is the Taylor-Proudman theorem.

Problem 2: Within the inertial range of a fully turbulent fluid, two particles start at a distance $\lambda_1 \gg \lambda_\circ$, where $\lambda_\circ$ is the viscous dissipation scale. At what time t are the particles separated by a distance λ_2, where $\lambda_1 \ll \lambda_2 \ll L$? Here, L is the outer scale of the bulk flow. Solve this problem two ways:

(a) Use dimensional analysis to estimate $d\lambda/dt$ in terms of λ itself and $\dot{\epsilon}$. Here, $\dot{\epsilon}$ is the (invariant) energy transfer rate per unit mass per unit time.

(b) At each scale λ, there is an effective viscosity $\nu_{\rm turb,\lambda}$:

$$\nu_{\rm turb,\lambda} \equiv u_\lambda\,\lambda \ ,$$

where u_λ is the appropriate eddy velocity. Recall that the ordinary kinematic viscosity ν is also a diffusion coefficient. Regarding the separation of the two points as occurring through diffusion, estimate again the time t and compare your answer to that of (a).

Problem 3: Consider the gravitational stability of an infinite, isothermal cylinder of gas. The equation of hydrostatic balance is

$$-a_T^2 \frac{d\rho}{dR} - \rho \frac{d\Phi}{dR} = 0 ,$$

where Φ is the gravitational potential. We conclude, as usual, that

$$\rho(R) = \rho_c \exp\left(-\Phi/a_T^2\right) ,$$

where ρ_c is the density at the central axis. Poisson's equation now reads

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) = 4\pi G \rho .$$

(a) Defining the nondimensional variables

$$\xi \equiv \frac{R\sqrt{4\pi\rho_c G}}{a_T}$$

$$\psi \equiv \frac{\Phi}{a_T^2} ,$$

find the equation for $\psi(\xi)$, as well as the appropriate boundary conditions.

(b) Find analytically $\psi(\xi)$, and thereby the density profile $\rho(R)$.

(c) Let M be the mass per unit length along the cylinder. The corresponding nondimensional quantity is

$$m \equiv \frac{G M}{a_T^2} .$$

Find analytically $m(\xi)$.

(d) To analyze stability, plot m as a function of ρ/ρ_e, the density contrast from center to edge. At what mass and density contrast does the stability transition occur?

Problem 4: C & C, Problem 36