AY 202
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Assignment 5 Solutions

① (a) Since the fluid is incompressible, mass conservation reads

$$\boxed{\vec{\nabla}\cdot\vec{u} = 0}$$

Momentum conservation, including both the Coriolis and centrifugal acceleration, is

$$\boxed{\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} + 2\vec{\Omega}\times\vec{u} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi}$$

(b) First prove the identity $\boxed{\frac{1}{2}\vec{\nabla}|\vec{\Omega}\times\vec{r}|^2 = -\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$

$$\begin{aligned}
|\vec{\Omega}\times\vec{r}|^2 = (\vec{\Omega}\times\vec{r})\cdot(\vec{\Omega}\times\vec{r}) &= \varepsilon_{ijk}\,\Omega_j x_k\,\varepsilon_{ilm}\,\Omega_l x_m \\
&= \Omega_j\Omega_l\,\varepsilon_{ijk}\varepsilon_{ilm}\,x_k x_m \\
&= \Omega_j\Omega_l\,(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\,x_k x_m \\
&= \Omega_j\Omega_j\,x_k x_k - \Omega_j x_j\,\Omega_k x_k
\end{aligned}$$

Thus, $\partial_i|\vec{\Omega}\times\vec{r}|^2 = 2\Omega_j\Omega_j x_i - 2\Omega_i\Omega_k x_k$ [using $\partial_i x_k = \delta_{ik}$]

or $\vec{\nabla}|\vec{\Omega}\times\vec{r}|^2 = 2\Omega^2\vec{r} - 2(\vec{r}\cdot\vec{\Omega})\vec{\Omega}$

Now $\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = (\vec{r}\cdot\vec{\Omega})\vec{\Omega} - \Omega^2\vec{r}$

Thus $\frac{1}{2}\vec{\nabla}|\vec{\Omega}\times\vec{r}|^2 = -\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ as was to be proved.

The momentum equation thus simplifies to

$$\boxed{\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} + 2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho}\vec{\nabla}p'}$$

where $p' \equiv p + \rho\Phi - \frac{1}{2}\rho|\vec{\Omega}\times\vec{r}|^2$

(c) For steady-state $\left(\frac{\partial}{\partial t} = 0\right)$, slow $\left[(\vec{u}\cdot\vec{\nabla})\vec{u} \approx 0\right]$ motion, we have

$$\boxed{2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho}\vec{\nabla}p'}$$

(d) Taking the curl of the above equation gives simply $\vec{\nabla}\times(\vec{\Omega}\times\vec{u}) = 0$

Thus, $\varepsilon_{ijk}\,\partial_j\,\varepsilon_{klm}\,\Omega_l u_m = \varepsilon_{kij}\,\varepsilon_{klm}\,\partial_j(\Omega_l u_m)$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\Omega_l\,\partial_j u_m$ since $\partial_j\Omega_l = 0$

$= \Omega_i\,\partial_j u_j - \Omega_l\,\partial_l u_i$ But $\partial_j u_j = \vec{\nabla}\cdot\vec{u} = 0$

We find $\Omega_l\,\partial_l u_i = 0 \longrightarrow \boxed{(\vec{\Omega}\cdot\vec{\nabla})\vec{u} = 0}$

② (a) Since $\dot{\epsilon}$ is the energy dissipation rate per mass per time, it has units of $\dot{\epsilon} \sim \frac{u^2}{t} = \frac{L^2}{t^3}$

So $\frac{d\lambda}{dt} \sim \frac{L}{t} = \left(\frac{L^2}{t^3} \cdot L\right)^{1/3} \longrightarrow \frac{d\lambda}{dt} = (\dot{\epsilon}\lambda)^{1/3}$ dimensionally

Integrating, $\frac{3}{2}\lambda^{2/3}\Big|_{\lambda_1}^{\lambda_2} \approx \frac{3}{2}\lambda_2^{2/3} = \dot{\epsilon}^{1/3} t$, since $\lambda_2 >> \lambda_1$

Dropping the numerical factor, we have the approximate result

$$\boxed{t = \left(\frac{\lambda_2^2}{\dot{\epsilon}}\right)^{1/3}}$$

(b) In a diffusion process, $\lambda \sim (\nu t)^{1/2}$. Here, ν is the diffusion coefficient, which is here $\nu_{turb,\lambda}$. Thus

$$t \sim \frac{\lambda^2}{\nu_{turb,\lambda}} = \frac{\lambda^2}{u_\lambda \lambda}$$

But in fully developed turbulence, $u_\lambda = (\dot{\epsilon}\lambda)^{1/3}$, so

$$t = \frac{\lambda^2}{\dot{\epsilon}^{1/3}\lambda^{4/3}} = \frac{\lambda^{2/3}}{\dot{\epsilon}^{1/3}} \longrightarrow \boxed{t = \left(\frac{\lambda_2^2}{\dot{\epsilon}}\right)^{1/3}} \text{ as before}$$

(3) (a) Combining Poisson's equation and hydrostatic balance yields

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) = 4\pi G\rho_c \exp\left(-\frac{\Phi}{a_T^2}\right)$$

The boundary conditions are $\Phi(0) = \Phi'(0) = 0$.

After making the change to nondimensional variables, we have

$$\boxed{\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\psi}{d\xi}\right) = e^{-\psi} \qquad \psi(0) = \psi'(0) = 0}$$

(b) Our equation may be recast as $\boxed{\frac{d^2\psi}{d\xi^2} + \frac{1}{\xi}\frac{d\psi}{d\xi} = e^{-\psi}}$

Let $\xi \equiv e^t$ | $\frac{d}{d\xi} = e^{-t}\frac{d}{dt}$; $\frac{d^2}{d\xi^2} = -e^{-2t}\frac{d}{dt} + e^{-2t}\frac{d^2}{dt^2}$

The equation becomes $\boxed{\frac{d^2\psi}{dt^2} = e^{2t-\psi}}$ Now let $y \equiv e^{2t-\psi} = \xi^2 e^{-\psi}$

We find $\boxed{\frac{d^2}{dt^2}\ln y + y = 0}$ Finally, let $z \equiv \ln y = 2\ln\xi - \psi$

The equation is $\frac{d^2 z}{dt^2} + e^z = 0$ Multiply through by $\frac{dz}{dt}$ –

$$\frac{dz}{dt}\frac{d^2z}{dt^2} + \frac{dz}{dt}e^z = 0 \longrightarrow \frac{1}{2}\frac{d}{dt}\left(\frac{dz}{dt}\right)^2 + \frac{d}{dt}\left(e^z\right) = 0$$ Integrating,

$\frac{1}{2}\left(\frac{dz}{dt}\right)^2 + e^z = C$, At $\xi = 0$ $(t = -\infty)$, $e^z = y = \xi^2 e^{-\psi} = 0$

Also, $dz/dt = 2 - d\psi/dt = 2 - \xi\, d\psi/d\xi = 2$

Thus $C = \frac{1}{2}(2)^2 = 2$

$$\frac{1}{2}\left(\frac{dz}{dt}\right)^2 + e^z = 2 \longrightarrow \frac{1}{2y^2}\left(\frac{dy}{dt}\right)^2 + y = 2 \longrightarrow \left(\frac{dy}{dt}\right)^2 + 2y^3 = 4y^2$$

$$\left(\frac{dy}{dt}\right)^2 = 2y^2(2-y) \longrightarrow \left(\frac{dy}{dt}\right) = \pm\sqrt{2}\, y\,(2-y)^{1/2}$$

This may be integrated: $\displaystyle\int\frac{dy}{y(2-y)^{1/2}} = \pm\sqrt{2}\,t + \text{const}$

The integral may be found in standard tables. We have

$$\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2} - \sqrt{2-y}}{\sqrt{2} + \sqrt{2-y}}\right) = \pm\sqrt{2}\,t + c = \pm\sqrt{2}\ln\xi + c$$

This equation may be solved for y:

$$y = 2\left[1 - \left(\frac{1 - c\xi^{\pm 2}}{1 + c\xi^{\pm 2}}\right)^2\right]$$

where I have generically used "c" to represent the constant

Recall that $y = \xi^2 e^{-\psi}$, which approaches ξ^2 as $\xi \to 0$ (since $e^{-\psi} \to 1$).

- First choose the $+$ sign for the exponent. Then, for small ξ,

$$\left(\frac{1 - c\xi^2}{1 + c\xi^2}\right)^2 \approx \left[(1 - c\xi^2)(1 - c\xi^2)\right]^2 \approx 1 - 4c\xi^2$$

Thus, $1 - \left(\frac{1 - c\xi^2}{1 + c\xi^2}\right)^2 \longrightarrow 4c\xi^2$. $\quad y \to 8c\xi^2$ and $\boxed{c = \frac{1}{8}}$

In this case, $\frac{\xi^2}{2} e^{-\psi} = 1 - \left(\frac{1 - \xi^2/8}{1 + \xi^2/8}\right)^2 = \frac{\xi^2/2}{(1 + \xi^2/8)^2}$

That is,

$$e^{-\psi} = \left(1 + \xi^2/8\right)^{-2} \longrightarrow \boxed{\frac{\rho}{\rho_c} = e^{-\psi} = \left(1 + \frac{\xi^2}{8}\right)^{-2}}$$

- Now choose the $-$ sign for the exponent. Then, for small ξ,

$$\frac{1 - c\xi^{-2}}{1 + c\xi^{-2}} = \frac{c\xi^{-2} - 1}{c\xi^{-2} + 1} = \frac{c - \xi^2}{c + \xi^2} = \frac{1 - \xi^2/c}{1 + \xi^2/c} \xrightarrow[\xi \to 0]{} 1 - 2\xi^2/c$$

Thus, $1 - \left(\frac{1 - c\xi^{-2}}{1 + c\xi^{-2}}\right)^2 \longrightarrow \frac{4\xi^2}{c}$ $\quad y \to \frac{8\xi^2}{c}$ and $\boxed{c = 8}$

In this case, $\frac{\xi^2}{2} e^{-\psi} = 1 - \left(\frac{1 - \xi^2/8}{1 + \xi^2/8}\right)^2$, just as before

(c) $M = \int_0^R 2\pi R \rho \, dR = \frac{2\pi a_T^2}{4\pi G} \int_0^{\xi_e} d\xi \, \xi e^{-\psi}$ where ξ_e is the (nondimensional) edge radius

Since $\frac{d}{d\xi}\left(\xi \frac{d\psi}{d\xi}\right) = \xi e^{-\psi}$

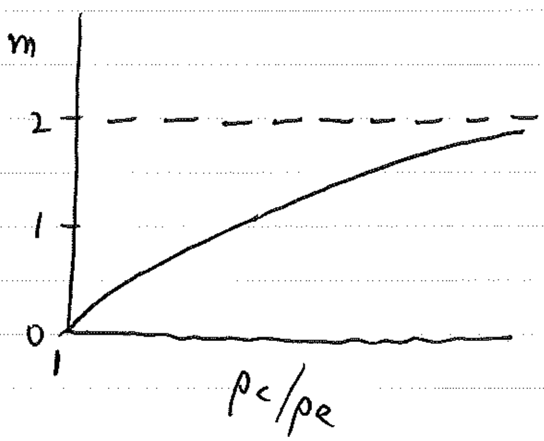
$$\int_0^{\xi_e} d\xi \, \xi e^{-\psi} = \left|\xi \frac{d\psi}{d\xi}\right|_{\xi_e} \qquad \text{Now } \frac{d\psi}{d\xi} = \frac{4\xi}{\xi^2 + 8}$$

So $M = \frac{2a_T^2}{G}\left[\frac{\xi_e^2}{\xi_e^2 + 8}\right] \longrightarrow \boxed{m = \frac{2\xi_e^2}{\xi_e^2 + 8}}$

(d) We need to replace ξ_e^2 by the density contrast ρ_c/ρ_e.

$$\frac{\rho_e}{\rho_c} = \left(1 + \frac{\xi_e^2}{8}\right)^{-2} \longrightarrow \xi_e^2 = 8\left(\sqrt{\frac{\rho_c}{\rho_e}} - 1\right)$$

Thus $$\boxed{m = \frac{2\left(\sqrt{\rho_c/\rho_e} - 1\right)}{\sqrt{\rho_c/\rho_e}}}$$



Unlike isothermal spheres, there is no turnover, and therefore no stability transition.

Dimensionally, if M is greater than $M_{crit} = \frac{2a_T^2}{G}$, no equilibrium is possible.

If $M < M_{crit}$, any equilibrium of any density contrast is stable.

C&C, problem 36

(4) (a) Let Γ, Λ be the heating and cooling rates per unit mass. We are given

$$\Gamma = \Gamma_0 \text{ (a constant)} \qquad \Lambda = k\rho T^{1/2}, \text{ where } k \text{ is a constant}$$

The Field criterion is that $\left(\frac{\partial \Lambda}{\partial T}\right)_p < 0$ for thermal instability

We know also that $P = \frac{\rho R T}{\mu} \rightarrow \rho = \frac{\mu P}{RT} \rightarrow \Lambda = \frac{k\mu P}{RT} T^{1/2}$

$$= \frac{k\mu P}{R} T^{-1/2}$$

$$\left(\frac{\partial \Lambda}{\partial T}\right)_p = -\frac{1}{2}\frac{\Lambda}{T} < 0 \quad \boxed{\text{Slab is thermally unstable}}$$

(b) In hydrostatic balance $\frac{dP}{dz} = -\rho \frac{d\psi}{dz}$

where $\psi(z)$ obeys Poisson's equation:

$$\frac{d^2\psi}{dz^2} = 4\pi G \rho$$

By symmetry, $d\psi/dz\,(0) = 0$. Since $\rho > 0$, $d^2\psi/dz^2 > 0$. Hence $d\psi/dz > 0$ for $z > 0$. The gravitational acceleration, $g = -d\psi/dz$, points downward, as shown.

In thermal balance, $\Gamma = \Lambda \rightarrow \Gamma_0 = k\rho T^{1/2}$ But $T = \frac{\mu P}{\rho R}$

Thus $\Gamma_0^2 = k^2\rho^2 T = k^2\rho^2 \frac{\mu P}{\rho R} = \frac{\mu k^2 P \rho}{R} \rightarrow P = \frac{1}{\rho}\frac{R\Gamma_0^2}{\mu k^2}$

Substituting $P(\rho)$ into the equation of hydrostatic balance yields

$$\frac{R\Gamma_0^2}{\mu k^2}\frac{1}{\rho^3}\frac{d\rho}{dz} = \frac{d\psi}{dz}$$

Thus $d\rho/dz > 0$. Since denser fluid overlies lighter fluid, the slab is $\boxed{\text{Rayleigh-Taylor unstable.}}$