AY 202 Assignment 6

due: Tuesday, April 19

Problem 1: We learned that a fluid is stable against convection if $ds/dz > 0$, where the coordinate z is in the direction of increasing gravitational potential. Here you will derive an analogous stability criterion for an axisymmetric, *rotating* fluid. Instead of $s(z)$, consider $j(R)$, where j is the angular momentum per unit mass and R is the radius outward from the rotation axis. Assume dynamic equilibrium, so that $f_{\rm cen}$, the centrifugal force per unit mass, is balanced at every R by some inward force $f_{\rm in}(R)$. (The latter would be stellar gravity in an astronomical gas, or the pressure gradient in a terrestrial fluid.)

(a) An element is initially located at $R = R_\circ$ and has specific angular momentum $j_\circ$. What is $f_{\rm cen}$ in terms of $j_\circ$ and $R_\circ$?

(b) The element is now displaced to a slightly larger radius R_1, where the equilibrium value of specific angular momentum is $j_1 \neq j_\circ$. If there are no torques during this motion, then the specific angular momentum of the moving fluid element is conserved. What is $f_{\rm out}$, the net *outward* force per unit mass acting on the element at R_1?

(c) By considering the limit of a very small displacement, derive the condition that must be satisfied by $j(R)$ for rotational stability of the fluid body.

(d) In Problem 1 of Assigment 3, you found $\Omega(R)$, the angular speed for fluid contained between two cylindrical walls of radius R_1 and R_2, and rotation speeds Ω_1 and Ω_2. Assuming the two walls rotate in the same direction, what condition must be satisfied by Ω_1 and Ω_2 for this motion to be stable at all R?

(e) Suppose the two cylindrical walls rotate in opposite directions. What is the condition for stability now?

(f) What is $j(R)$ in a low-mass disk orbiting a star of mass M_*? Is the disk stable or unstable?

Problem 2: The rate of mass infall onto the central object of a collapsing, self-gravitating gas cloud is given approximately by

$$\dot{M} = \frac{a_T^3}{G} ,$$

where a_T is the isothermal sound speed.

(a) In class, we derived this relation by arguing that a rarefaction wave spreads out from the accreting central object. Alternatively, show that the relation follows from the fact that the parent body contains about a Jeans mass and that it collapses over a free-fall time.

(b) Another key concept in accretion physics is the Eddington luminosity. This is defined to be

$$L_E \equiv \frac{4\,\pi\,c\,G\,M}{\kappa} \; .$$

Here, M is the central object's mass, while κ is the opacity of infalling gas. Show that L_E is the maximum luminosity that can be emitted by the object before radiation pressure reverses infall.

(c) Now apply these concepts to star formation. The luminosity of a low-mass protostar of mass M_* and radius R_* is that released by accretion onto the stellar surface:

$$L_{\rm acc} = \frac{G\,M\,\dot{M}}{R_*} \; .$$

Consider such a protostar forming out of a dusty molecular cloud with $T = 15$ K. Suppose further that the protostar's central temperature is 1×10^6 K. This latter fact, plus the virial theorem, gives you a relationship between M_* and R_*. Using $\kappa = 10\ {\rm cm}^{-2}\,{\rm g}^{-1}$, appropriate for near-infrared radiation impinging on interstellar grains, estimate the ratio $L_E/L_{\rm acc}$. Is radiation pressure important here?

(d) Finally, consider a protostar with enough mass that it is fusing hydrogen while still accreting. The protostar's luminosity is then given, not by $L_{\rm acc}$, but by the main-sequence value, $L_{\rm ms}$, at the mass in question. By consulting a table of main-sequence luminosities, determine the critical protostellar mass, $M_{\rm crit}$, where radiation pressure first becomes significant.

Problem 3: The winds from massive stars are accelerated because their intense ultraviolet radiation is absorbed by spectral lines in ions. The momentum equation for such a line-driven wind, assuming isothermality and spherical symmetry, is

$$u\,\frac{du}{dr} = -\frac{G\,M_*}{r^2} + f_{\rm rad} \; ,$$

where we have dropped the gas pressure term because we are in the supersonic regime. The radiative force per unit mass, $f_{\rm rad}$, is given approximately by

$$f_{\rm rad} = \frac{\kappa_\circ\,F_\circ\,\Delta\nu_D}{c\,\tau} \; .$$

Here, $\kappa_\circ$ is the opacity at line center, $F_\circ$ is the continuum specific flux from the star (in ergs ${\rm cm}^{-2}\,{\rm s}^{-1}\,{\rm Hz}^{-1}$), also evaluated at line center, and $\Delta\nu_D$ is the Doppler width of the line. The optical thickness τ is given in terms of the wind density and velocity gradient by

$$\tau = \rho \, \kappa_\circ \, \Delta V_{\rm therm} \, (du/dr)^{-1} \ .$$

where $\Delta V_{\rm therm}$ is the velocity dispersion arising from thermal motion.

Taking $\kappa_\circ$, $\Delta\nu_D$, and $\Delta V_{\rm therm}$ to be constants, derive an analytic expression for $u(r)$, valid as the speed approaches its asymptotic, terminal value, u_∞. In your expression, let $\dot{M}_w$ be the total wind mass loss rate, and F_* the continuum specific flux at the stellar radius, $r = R_*$.

Problem 4: C & C, Problem 49. Note that the quantity S defined in the problem should have a factor of 1/3 in front.