AY 202
Stahler

Assignment 6 Solutions

① (a) The centrifugal force per unit mass is u_φ^2/R. Since $j = R u_\varphi$,

$$\boxed{f_{cen} = j^2/R_0^3}$$

(b) The new outward force is $f_{cen}(R_1) = j_0^2/R_1^3$. Here I have used the fact that j is conserved during the displacement. There is also an inward force, $f_{in}(R_1)$. This used to balance the old centrifugal force, prior to the displacement. That is, $f_{in}(R_1) = j_1^2/R_1^3$. So the net outward force is

$$\boxed{f_{out} = \frac{j_0^2 - j_1^2}{R_1^3}}$$

(c) For stability, $f_{out} < 0$, or $j_0^2 < j_1^2$. Remember that $j_0 \equiv j(R_0)$ and $j_1 \equiv j(R_1)$. Thus, in the limit of small displacement,

$$\boxed{\frac{\partial j^2}{\partial R} > 0 \quad \text{for stability}}$$

(d) We found

$$u_\varphi(R) = \frac{R_2^2 \Omega_2 - R_1^2 \Omega_1}{(R_2^2 - R_1^2)} R + \frac{R_1^2 R_2^2 (\Omega_1 - \Omega_2)}{(R_2^2 - R_1^2) R},$$

where Ω_1 and R_1 refer to the inner cylindrical wall, etc. Thus,

$$j(R) = R u_\varphi = \frac{R_2^2 \Omega_2 - R_1^2 \Omega_1}{(R_2^2 - R_1^2)} R^2 + \frac{R_1^2 R_2^2 (\Omega_1 - \Omega_2)}{(R_2^2 - R_1^2)}$$

$$\equiv AR^2 + B$$

Thus, $j^2 = A^2 R^4 + B^2 + 2ABR^2$. For stability

$$\frac{\partial j^2}{\partial R} = 4A^2 R^3 + 4ABR > 0$$

Suppose first that $A > 0$.* Then, since $4AR > 0$, the above inequality is equivalent to $AR^2 + B > 0$. But $AR^2 + B > AR_1^2 + B = j(R_1) = R_1^2 \Omega_1 > 0$. So the flow is indeed stable.

* and that both Ω_1 and Ω_2 are positive numbers.

Now suppose that $A<0$. Since $4AR<0$, we need $AR^2+B<0$. But, since $A<0$, $AR^2+B > AR_2^2+B = j(R_2) = R_2^2\Omega_2 > 0$. Thus, the flow is unstable.

In summary, stability requires that $A>0$, or $\boxed{\Omega_2 > \Omega_1 (R_1^2/R_2^2)}$

(e) Let $\Omega_1>0$ and $\Omega_2<0$. Then $A<0$ and $B>0$. We still require $4A^2R^3 + 4ABR > 0$ for stability. The first term is positive, the second negative. We need

$$4A^2R^3 > -4ABR \quad \text{or}$$

$$|A|R^2 > B \quad \text{which is}$$

$$\frac{R_1^2\Omega_1 - R_2^2\Omega_2}{(R_2^2-R_1^2)} R^2 > \frac{R_1^2R_2^2(\Omega_1-\Omega_2)}{(R_2^2-R_1^2)}$$

$$(R_1^2\Omega_1 - R_2^2\Omega_2)R^2 > R_1^2R_2^2(\Omega_1-\Omega_2)$$

When $R=R_1$, the lefthand side is $R_1^4\Omega_1 - R_1^2R_2^2\Omega_2 < R_1^2R_2^2\Omega_1 - R_1^2R_2^2\Omega_2$
When $R=R_2$, the lefthand side is $R_1^2R_2^2\Omega_1 - R_2^4\Omega_2 > R_1^2R_2^2\Omega_1 - R_1^2R_2^2\Omega_2$
There is thus a critical radius where $\partial j^2/\partial R = 0$. This radius is given by

$$(R_1^2\Omega_1 - R_2^2\Omega_2)R_{crit}^2 = R_1^2R_2^2(\Omega_1-\Omega_2)$$

$$\text{or} \quad R_{crit}^2 = \frac{R_1^2R_2^2(\Omega_1-\Omega_2)}{R_1^2\Omega_1 - R_2^2\Omega_2} = R_2^2 f + R_1^2(1-f)$$

where $f \equiv \dfrac{R_1^2\Omega_1}{R_1^2\Omega_1 - R_2^2\Omega_2}$ (a number between 0 and 1)

A similar analysis holds if $\Omega_1<0$ and $\Omega_2>0$. In summary, the flow is unstable.

(f) Here, $u_\varphi^2 = GM_*/R \rightarrow j^2 = GM_*R$. Since $\partial j^2/\partial R = GM_* > 0$, the flow is stable.

② (a) $M_J \sim \frac{a_T^3}{\rho^{1/2} G^{3/2}}$ $t_{ff} \sim \frac{1}{G^{1/2}\rho^{1/2}}$

$$\dot{M} \sim \frac{M_J}{t_{ff}} \sim a_T^3 \rho^{-1/2} G^{-3/2} G^{1/2} \rho^{1/2} = \frac{a_T^3}{G} \checkmark$$

(b) The momentum per unit area per unit time carried by radiation is F_{rad}/c. Thus, the radiative force per unit mass on a gas element is $\kappa F_{rad}/c$, where the opacity κ has units of $cm^2\ gm^{-1}$.

This radiative force equals gravity when $\frac{\kappa F_{rad}}{c} = \frac{GM}{r^2}$

Setting $F_{rad} = L/4\pi r^2$, we find $\boxed{L = \frac{4\pi c G M}{\kappa}}$

(c) $L_{acc} = \frac{G M \dot{M}}{R_*} \approx \frac{M a_T^3}{R_*}$ Here, a_T is the isothermal sound speed <u>in the cloud</u> $= 0.2$ km s^{-1} for $T = 1$ K.

From the virial theorem,

$$\frac{GM}{R_*} \approx \mathcal{R} T_c$$

Here, T_c is the central temperature of the protostar, given as 1×10^6 K.

Thus, $L_{acc} = \frac{\mathcal{R} T_c a_T^3}{G} = 2\ L_\odot$

However, $L_E = \frac{4\pi c G M}{\kappa} = 1 \times 10^3\ L_\odot$ for $M = 1\ M_\odot$

Thus, $L_E / L_{acc} \sim 10^3$ for solar-mass stars, and we can <u>ignore</u> radiation pressure.

(d) Using Table 1.1 of "The Formation of Stars" for L_{MS}:

$M/M_\odot =$	$L_{MS}/L_\odot$	$L_E/L_\odot$
6	8×10^2	7×10^3
8	2×10^3	1×10^4
10	6×10^3	1×10^4
18	5×10^4	2×10^4

Radiation pressure becomes important for $\boxed{M_* \gtrsim 10\ M_\odot}$

(3) In the supersonic regime, we may ignore the gas pressure gradient. The momentum equation becomes:

$$u\frac{du}{dr} = -\frac{GM_*}{r^2} + \frac{k_0 F_0 \Delta V_D}{c\tau}$$

$$= -\frac{GM_*}{r^2} + \frac{F_0 \Delta V_D}{\rho c \Delta V_{therm}}\frac{du}{dr}$$

The continuum specific flux is $F_0 = F_*\left(\frac{R_*}{r}\right)^2$, where F_* and R_* are the flux and radius, respectively at the stellar surface. Thus

$$u\frac{du}{dr} = -\frac{GM_*}{r^2} + \frac{F_* R_*^2 \Delta V_D}{\rho c r^2 \Delta V_{therm}}\frac{du}{dr}$$

From mass continuity, $4\pi r^2 \rho u = \dot{M}_W \rightarrow \rho r^2 = \frac{\dot{M}_W}{4\pi u}$

Thus $$u\frac{du}{dr} = -\frac{GM_*}{r^2} + \frac{F_* R_*^2 \Delta V_D}{c\dot{M}_W \Delta V_{therm}} 4\pi u \frac{du}{dr}$$

Let $$\alpha \equiv \frac{4\pi F_* R_*^2 \Delta V_D}{\dot{M}_W c \Delta V_{therm}} - 1$$ (a nondimensional quantity)

$$u\frac{du}{dr} = +\frac{GM_*}{r^2\alpha}$$

Clearly, α must be positive so that the wind accelerates outward.

Integrating the above equation from r to ∞ (where $u = u_\infty$),

$$\frac{1}{2}(u_\infty^2 - u^2) = \frac{GM_*}{\alpha}\left|-\frac{1}{r}\right|_r^\infty = \frac{GM_*}{\alpha r}$$

We find that $$\boxed{u = u_\infty\sqrt{1 - \frac{2GM_*}{\alpha u_\infty^2 r}}}$$

C&C, Problem 49

(4) The evolution equation derived in class is

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)\right]$$

Let $S \equiv \frac{1}{3}\Sigma R^{3/2}$ (the book omits the factor of $1/3$)

$y \equiv \nu \Sigma R^{1/2}$

$X \equiv 2R^{1/2} \longrightarrow R = \frac{X^2}{4} \longrightarrow \frac{dR}{dX} = \frac{X}{2} = R^{1/2}$

Thus $\frac{\partial}{\partial X} = \frac{dR}{dX}\frac{\partial}{\partial R} = R^{1/2}\frac{\partial}{\partial R}$

We have $\frac{\partial \Sigma}{\partial t} = 3R^{-3/2}\frac{\partial S}{\partial t}$, and $R^{1/2}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right) = \frac{\partial y}{\partial X}$.

So the evolution equation becomes $3R^{-3/2}\frac{\partial S}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\frac{\partial y}{\partial X}$

or $\boxed{\frac{\partial S}{\partial t} = R^{1/2}\frac{\partial}{\partial R}\frac{\partial y}{\partial X} = \frac{\partial^2 y}{\partial X^2}}$

If $y = y_0 + \beta(S - S_0)$, then $\frac{\partial^2 y}{\partial X^2} = \beta\frac{\partial^2 S}{\partial X^2}$, and

$$\boxed{\frac{\partial S}{\partial t} = \beta\frac{\partial^2 S}{\partial X^2}}$$

If $\beta > 0$ [$\nu\Sigma R^{1/2}$ increases with $R^{1/2}$], $S(X,t)$ diffuses outward.

If $\beta < 0$ [$\nu\Sigma R^{1/2}$ decreases with $R^{1/2}$] $S(X,t)$ concentrates inward.