Assignment 7 Solutions

① (a) Hydrostatic balance is: $\boxed{\dfrac{1}{\rho}\dfrac{dp}{dr} = -\dfrac{GM_\odot}{r^2}}$

(b) Since $p = \rho a_T^2$, $\quad a_T^2 \dfrac{d\ln p}{dr} = -\dfrac{GM_\odot}{r^2}$

Integrating from r_0 to ∞, where $p = p_\infty = \rho_\infty a_T^2$,

$$\rho_\infty = \rho_0 \exp\left[-\frac{GM_\odot}{a_T^2 r_0}\right] \rightarrow \boxed{p_\infty = p_0 \exp\left[-\frac{GM_\odot}{a_T^2 r_0}\right]}$$

(c) $p = n_{tot} k_B T$ where n_{tot} is the number density of particles

At $r = r_0$, the number density of H atoms is $n_H = 3 \times 10^7\ \mathrm{cm}^{-3}$. Assuming a pure hydrogen gas, $n_{tot} = 6 \times 10^7\ \mathrm{cm}^{-3}$ (since the gas is ionized). Using $T = 2 \times 10^6$ K,

$$\boxed{p_0 = 1.7 \times 10^{-2}\ \mathrm{dyne/cm^2}}$$

To find p_∞, we need $\dfrac{GM_\odot}{a_T^2 r_0}$. Use $a_T^2 = \dfrac{\mathcal{R}T}{\mu} = 2\mathcal{R}T$

For $T = 2 \times 10^6$ K, $r_0 = 1.25\ R_\odot$: $\quad \dfrac{GM_\odot}{a_T^2 r_0} = 4.59 \rightarrow \boxed{p_\infty = 1.7 \times 10^{-4}\ \mathrm{dyne/cm^2}}$

(d) In the (neutral) HI cloud, $n_{tot} = n_H = 30\ \mathrm{cm}^{-3}$
Using $T = 80$ K,

$$\boxed{p_{ism} = 3.3 \times 10^{-13}\ \mathrm{dyne/cm^2}}$$

Since $p_\infty \gg p_{ism}$, the hypothetical static envelope would immediately expand — a wind!

(2) (a) The ideal MHD equation in steady state is

$$\frac{\partial \vec{B}}{\partial t} = \boxed{0 = \vec{\nabla} \times (\vec{u} \times \vec{B})}$$

(b) The linear version of the above equation is $\boxed{0 = \vec{\nabla} \times (\vec{u}' \times \vec{B}_0)}$

Mass continuity in steady state is $0 = \vec{\nabla} \cdot (\rho \vec{u})$

The linearized version is $0 = \vec{\nabla} \cdot (\rho_0 \vec{u}')$. Since ρ_0 is spatially constant,

$$\boxed{0 = \vec{\nabla} \cdot \vec{u}'}$$

In index notation, the i^{th} component of ideal MHD is

$$\left[\vec{\nabla} \times (\vec{u}' \times \vec{B}_0)\right]_i = \varepsilon_{ijk} \partial_j \varepsilon_{klm} u_l' B_{0m} = \varepsilon_{kij} \varepsilon_{klm} \partial_j (u_l' B_{0m})$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(B_{0m}\partial_j u_l' + u_l' \partial_j B_{0m})$$

$$= B_{0j}\partial_j u_i' + u_i' \partial_j B_{0j} - B_{0i}\partial_j u_j' - u_j' \partial_j B_{0i}$$

In ordinary vector notation,

$$\vec{\nabla} \times (\vec{u}' \times \vec{B}_0) = (\vec{B}_0 \cdot \vec{\nabla})\vec{u}' + \vec{u}'(\vec{\nabla} \cdot \vec{B}_0) - \vec{B}_0(\vec{\nabla} \cdot \vec{u}') - (\vec{u}' \cdot \vec{\nabla})\vec{B}_0$$

Now, $\vec{\nabla} \cdot \vec{B}_0 = \vec{\nabla} \cdot \vec{u}' = 0$. Since $\vec{B}_0$ is spatially constant, $(\vec{u}' \cdot \vec{\nabla})\vec{B}_0 = 0$

Hence $\vec{\nabla} \times (\vec{u}' \times \vec{B}_0) = \boxed{0 = (\vec{B}_0 \cdot \vec{\nabla})\vec{u}'}$

The velocity does not change along the direction of $\vec{B}_0$. This is indeed the analog of the Taylor-Proudman theorem, which states that $\vec{u}'$ does not change along the direction of the angular velocity vector $\vec{\Omega}$.

③ (a) Ideal MHD again implies $\vec{\nabla} \times (\vec{u} \times \vec{B}) = 0$

Write this as $\vec{\nabla} \times \vec{C} = 0$, where $\vec{C} \equiv \vec{u} \times \vec{B}$

In cylindrical coordinates

$$\begin{aligned}\vec{C} = &(u_\varphi B_z - u_z B_\varphi)\hat{e}_R \rightarrow u_\varphi B_z \hat{e}_R \quad (\text{since } u_z = B_\varphi = 0)\\ + &(u_z B_R - u_R B_z)\hat{e}_\varphi \rightarrow 0 \quad (\text{since } u_z = u_R = 0)\\ + &(u_R B_\varphi - u_\varphi B_R)\hat{e}_z \rightarrow -u_\varphi B_R \hat{e}_z \quad (\text{since } B_\varphi = 0)\end{aligned}$$

Now $(\vec{\nabla} \times \vec{C})_\varphi = \dfrac{\partial C_R}{\partial z} - \dfrac{\partial C_z}{\partial R} = \dfrac{\partial (u_\varphi B_z)}{\partial z} + \dfrac{\partial (u_\varphi B_R)}{\partial R}$

$$= u_\varphi \frac{\partial B_z}{\partial z} + B_z \frac{\partial u_\varphi}{\partial z} + u_\varphi \frac{\partial B_R}{\partial R} + B_R \frac{\partial u_\varphi}{\partial R}$$

Also, $0 = \vec{\nabla} \cdot \vec{B} = \dfrac{1}{R}\dfrac{\partial}{\partial R}(R B_R) + \dfrac{\partial B_z}{\partial z} = \dfrac{\partial B_R}{\partial R} + \dfrac{B_R}{R} + \dfrac{\partial B_z}{\partial z}$

Thus $(\vec{\nabla} \times \vec{C})_\varphi = u_\varphi \dfrac{\partial B_z}{\partial z} + B_z \dfrac{\partial u_\varphi}{\partial z} + u_\varphi\left(-\dfrac{B_R}{R} - \dfrac{\partial B_z}{\partial z}\right) + B_R \dfrac{\partial u_\varphi}{\partial R}$

$$= B_z \frac{\partial u_\varphi}{\partial z} + B_R \frac{\partial u_\varphi}{\partial R} - B_R \frac{u_\varphi}{R}$$

$$= R B_z \frac{\partial (u_\varphi / R)}{\partial z} + R B_R \frac{\partial (u_\varphi / R)}{\partial R}$$

$$= R(\vec{B} \cdot \vec{\nabla})(u_\varphi / R)$$

We thus find $\boxed{(\vec{B} \cdot \vec{\nabla})(u_\varphi / R) = 0}$

(b) The angular speed, $\Omega = u_\varphi / R$, does not change along the poloidal field $\vec{B}$. The field lines rotate rigidly. If Ω did change, the differential rotation would wrap up $\vec{B}$, creating a toroidal component, B_φ.

C&C Problem 54

(4) (a) Hydrostatic equilibrium is: $0 = -\vec{\nabla} p + \frac{\vec{J}}{c} \times \vec{B}$

Using Ampère's law: $\frac{\vec{J}}{c} = \frac{1}{4\pi} \vec{\nabla} \times \vec{B}$, we find

$$\vec{\nabla} p = \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \times \vec{B}$$

Now
$$\left[(\vec{\nabla} \times \vec{B}) \times \vec{B}\right]_R = (\vec{\nabla} \times \vec{B})_\varphi B_z - (\vec{\nabla} \times \vec{B})_z \cancelto{0}{B_\varphi}$$
$$= \left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right) B_z = -B_z \frac{\partial B_z}{\partial R}$$

So
$$\frac{\partial p}{\partial R} = -\frac{1}{4\pi} B \frac{\partial B}{\partial R} = -\frac{1}{8\pi} \frac{\partial B^2}{\partial R}$$

or
$$\boxed{\underbrace{p}_{\text{gas pressure}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic pressure}} = \text{constant}}$$

(b) Let p_0 be the gas pressure outside the flux tube, where $B = 0$. Then

$$\Delta p = p_0 - p_{in} = \frac{B^2}{8\pi}$$

The flux tube significantly affects its surroundings if $\frac{\Delta p}{p_0} \sim 1$. Here,

$$\frac{\Delta p}{p_0} = \frac{B^2}{8\pi p_0},$$ where $p_0 = \frac{\rho R T}{\mu}$. Using $\rho = 5 \times 10^{-7}$ g cm^{-3}, $T = 4000$ K, $\mu = 1$

$$= 2 \times 10^5 \frac{\text{dyne}}{\text{cm}^2}$$

Now $B = 10^3$ G $(= 1$ kG$)$, so $\frac{B^2}{8\pi} = 4 \times 10^4$ dyne/cm^2

Thus, $\boxed{\frac{\Delta p}{p_0} = 0.2}$, and the flux tube <u>does</u> affect its surroundings.