AY 202
Stahler

Final Exam Solutions

① Since the motion is subsonic, we ignore acceleration in the momentum equation, which just becomes the condition of hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM}{r^2}$$

Here, M is the mass of the white dwarf. Since the settling is adiabatic,

$$P = k\rho^{5/3} = \frac{\rho \mathcal{R} T}{\mu}$$

Thus, $T = \frac{\mu k}{\mathcal{R}}\rho^{2/3}$ and $k = \frac{\mathcal{R}T_\infty}{\mu \rho_\infty^{2/3}}$ where T_∞ and ρ_∞ are the temperature and density far from the white dwarf surface.

To find $\rho(r)$ [and thereby $T(r)$], use

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{1}{\rho}\frac{5}{3}k\rho^{2/3}\frac{d\rho}{dr} = \frac{5}{3}k\rho^{-1/3}\frac{d\rho}{dr} = -\frac{GM}{r^2}$$

Integrate from the inner density ρ_{in} at $r = r_{in}$ to the final density ρ_∞ at $r = \infty$:

$$\frac{5}{3}k\frac{3}{2}\left|\rho^{2/3}\right|_{\rho_{in}}^{\rho_\infty} = GM\left(\frac{1}{r}\right)_{r_{in}}^{\infty}$$

$$\frac{5}{2}k\left(\rho_\infty^{2/3} - \rho_{in}^{2/3}\right) = -\frac{GM}{r_{in}}$$

$$\frac{5}{2}k\left[\left(\frac{\rho_{in}}{\rho_\infty}\right)^{2/3} - 1\right] = \frac{GM}{r_{in}\rho_\infty^{2/3}}$$

$$\frac{T_{in}}{T_\infty} = \left(\frac{\rho_{in}}{\rho_\infty}\right)^{2/3} = 1 + \frac{2GM}{5kr_{in}\rho_\infty^{2/3}}$$

$$\boxed{\frac{T_{in}}{T_\infty} = 1 + \frac{2GM\mu}{5\mathcal{R}T_\infty r_{in}}}$$

Using $M = 0.5\,M_\odot$, $T_\infty = 2\times10^7\,K$, and $r_{in} = 0.01\,R_\odot$ (and $\mu = 0.5$)

$$\frac{T_{in}}{T_\infty} = 12 \longrightarrow \boxed{T_{in} = 2.4\times10^8\,K}$$

(2) (a) In a perfectly conducting fluid, the ideal MHD equation applies:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) \qquad \text{or}$$

$$\frac{\partial B_i}{\partial t} = \varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm}\, u_l B_m$$

$$= \varepsilon_{kij}\, \varepsilon_{klm}\, \partial_j (u_l B_m)$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, \partial_j (u_l B_m)$$

$$= \partial_j (u_i B_j) - \partial_j (u_j B_i)$$

$$= u_i \partial_j B_j + B_j \partial_j u_i - B_i \partial_j u_j - u_j \partial_j B_i$$

Thus $\frac{\partial \vec{B}}{\partial t} = \underbrace{(\vec{\nabla} \cdot \vec{B})}_{0}\vec{u} + (\vec{B} \cdot \vec{\nabla})\vec{u} - (\vec{\nabla} \cdot \vec{u})\vec{B} - (\vec{u} \cdot \vec{\nabla})\vec{B}$

$$\boxed{\frac{D\vec{B}}{Dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{B} = (\vec{B} \cdot \vec{\nabla})\vec{u} - (\vec{\nabla} \cdot \vec{u})\vec{B}}$$

(b) The Lagrangian form of mass continuity is

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -(\vec{\nabla} \cdot \vec{u})$$

$$\frac{D\vec{B}}{Dt} = (\vec{B} \cdot \vec{\nabla})\vec{u} + \frac{1}{\rho}\frac{D\rho}{Dt}\vec{B}$$

$$\frac{1}{\rho}\frac{D\vec{B}}{Dt} - \frac{1}{\rho^2}\frac{D\rho}{Dt}\vec{B} = \left(\frac{\vec{B}}{\rho} \cdot \vec{\nabla}\right)\vec{u}$$

$$\frac{1}{\rho}\frac{D\vec{B}}{Dt} + \vec{B}\frac{D(1/\rho)}{Dt} = \boxed{\frac{D(\vec{B}/\rho)}{Dt} = \left(\frac{\vec{B}}{\rho} \cdot \vec{\nabla}\right)\vec{u}}$$

(c) As long as $\vec{u}$ changes in the direction of $\vec{B}$, $D(\vec{B}/\rho)/Dt$ is <u>not</u> zero.