Star Formation: Problem set 1

N. Bremer

September 9, 2010

Problem 1

a. The filament is faced nearly face on, thus we are looking through an edge-on circle. The volumetric density n_H can be found by dividing the given column density N_H by the average length of the filament that we see. The filament has the shape of a circle, thus the distance that is looked though, D, is not the same at each location. To find the average distance, $\langle D \rangle$, we use the fact that we can make a right triangle between the radius of the circle, R, half of the look-through distance D and the part of the triangle perpendicular to the line of sight, which we call x. The relation between the three is then $D(x) = 2\sqrt{R^2 - x^2}$. The average of D can be found by integrating over x between 0 and R (or equivalently between $-R$ and R.)

$$
\begin{aligned}
\langle D \rangle &= \frac{\int_0^R D(x)\,\mathrm{d}x}{\int_0^R \mathrm{d}x} = \frac{\int_0^R 2\sqrt{R^2 - x^2}\,\mathrm{d}x}{[x]_0^R} \\
&= \frac{\int_0^R 2\sqrt{R^2 - x^2}\,\mathrm{d}x}{R} \\
&= 2\int_0^R \sqrt{1 - x^2/R^2}\,\mathrm{d}x \\
&= 2R\int_0^1 \sqrt{1 - y^2}\,\mathrm{d}y \\
&= 2R\int_0^{\pi/2} \sqrt{1 - \sin^2\theta}\cos\theta\,\mathrm{d}\theta \\
&= 2R\int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta \\
&= 2R\int_0^{\pi/2} \frac{1}{2} + \frac{\cos 2\theta}{2}\,\mathrm{d}\theta \\
&= 2R\left[\frac{1}{2}\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2} \\
&= \frac{\pi R}{2}\,. \qquad (1)
\end{aligned}
$$

In the above sequence of equations two substitutions were used. The first was $y \equiv x/R$ and the second was $y \equiv \sin\theta$.
For this filament $R = 2$ pc, thus $\langle D\rangle = 9.69 \cdot 10^{18}$ cm

$$n_H = \frac{N_H}{\langle D\rangle} = \frac{2 \cdot 10^{22}\,\mathrm{cm}^{-2}}{9.69 \cdot 10^{18}\,\mathrm{cm}} = 2.06 \cdot 10^{3}\,\mathrm{cm}^{-3}\,. \tag{2}$$

b. The total mass of the filament can be found by multiplying the average density ρ by the total volume V of the filament. Here ρ is given by

$$\rho = m_H n_H \mu\,, \tag{3}$$

where m_H is the hydrogen mass of $1.67 \cdot 10^{-24}$ g and μ is the mean molecular weight. For a molecular gas of Solar composition $\mu = 2.4$. Using n_H from question a; $\rho = 8.27 \cdot 10^{-21}$ g/cm^{-3}. The volume of the filament is given by

$$V = \pi \left(\frac{\langle D\rangle}{2}\right)^2 L = 3.64 \cdot 10^{57}\,\mathrm{cm}^3\,, \tag{4}$$

where D is the diameter of 4 pc and L is its length of 16 pc. The total mass of the filament is then

$$\begin{aligned} M_{\mathrm{total}} &= \rho V = 8.72 \cdot 10^{-21}\,\mathrm{g/cm}^{-3} \cdot 3.64 \cdot 10^{57}\,\mathrm{cm}^3 \\ &= 3.01 \cdot 10^{37}\,\mathrm{g} = 1.51 \cdot 10^{4}\,\mathrm{M}_\odot\,. \end{aligned} \tag{5}$$

c. The line width of ^{13}C^{16}O has the same speed as the stars. The virial speed, $v_{\mathrm{vir}} = \sqrt{GM/R}$, of the filament is also of the same order. This is evidence for the fact that the stars are not on ballistic trajectories, but are experiencing random motions together with the gas. It is thus expected that over the typical age of the stars, the filamentary pattern will be maintained.

Problem 2

a. The color excess, $E_{\mathrm{B-V}}$, is given by

$$\begin{aligned} E_{\mathrm{B-V}} &= (m_B - m_V) - (M_B - M_V) \\ &= (14.3 - 12.8) - (0 - 0) = 1.5\,, \end{aligned} \tag{6}$$

where m_B amd m_V are the apparent magnitudes in the B- and V-band, respectively, and M_B and M_V are the absolute mangitudes in the B- and V-band, respectively. By definition, an A0 star has zero absolute magnitude in all wavebands.

b. Using formula 2.16 from the book, A_V is given by

$$A_V = R\,E_{\mathrm{B-V}} = 3.1 \cdot 1.5 = 4.65\,. \tag{7}$$

c. Using formula 3.2 from the book, N_H is given by

$$N_H = A_V/5.3 \cdot 10^{-22}\,\mathrm{mag\,cm^2} = 8.77 \cdot 10^{21}\,\mathrm{cm^{-2}}\,. \tag{8}$$

d. Starting from formula 2.15b from the book

$$\frac{E_{\lambda-\mathrm{V}}}{E_{\mathrm{B-V}}} = \frac{A_\lambda}{E_{\mathrm{B-V}}} - R\,. \tag{9}$$

This can be rewritten to

$$A_\lambda = E_{\mathrm{B-V}} \left(\frac{E_{\lambda-\mathrm{V}}}{E_{\mathrm{B-V}}} + R \right)\,. \tag{10}$$

The value of $E_{\lambda-\mathrm{V}}/E_{\mathrm{B-V}}$ for the K-band can be found in figure 2.7 in the book. The K-band is centered on 2.2 μm, thus $\lambda^{-1} = 1/2.2 = 0.45\,\mu\mathrm{m}^{-1}$. The value of the extinction curve at 0.45 is ~ -3. Together with the previously calculated value for the color excess

$$A_K = 1.5\,(-3 + 3.1) = 0.15\,. \tag{11}$$

The factor by which the K-band flux from the star is attenuated can be found by looking at formula 2.22 from the book an seeing that $F_\lambda \propto \exp(-\Delta\tau_\lambda)$. The following relation between A_λ and $\Delta\tau_\lambda$ can be used

$$A_\lambda = 1.086\,\Delta\tau_\lambda\,. \tag{12}$$

Thus the K-band flux is attenuated by a factor of

$$\exp(-\Delta\tau_K) = \exp(-A_K/1.086) = 0.87\,. \tag{13}$$

Problem 3

a. Starting from $R \equiv A_V/E_{\mathrm{B-V}}$, this equation can be rewritten by remembering the fact that you can also write $E_{\mathrm{B-V}}$ as $A_B - A_V$. Thus $R \equiv A_V/(A_B - A_V)$. Furthermore, from formula 2.26 in the book we see that $A_\lambda \propto \Delta\tau_\lambda$ and finally we know that $\tau_\lambda \propto \kappa_\lambda$. This implies that

$$R = \frac{A_V}{A_B - A_V} = \frac{\kappa_V}{\kappa_B - \kappa_V} = \frac{\lambda_V^{-n}}{\lambda_B^{-n} - \lambda_V^{-n}}\,, \tag{14}$$

where the fact that $\kappa_\lambda \propto \lambda^{-n}$ was used. This can also be compactly written by dividing both the nominator and denominator by λ_V^{-n}

$$R = \frac{1}{(\lambda_B/\lambda_V)^{-n} - 1}\,. \tag{15}$$

The B-band is centered on 4400 Å and the V-Band on 5500 Å. Thus

$$R = \frac{1}{(4400/5500)^{-n} - 1} = \left(0.8^{-n} - 1\right)^{-1}\,. \tag{16}$$

b. Using formula 16 and the fact that $R = 3.1$ the value of n can be found numerically to be $n = 1.25$. Comparing this to figure 2.7 in the book the slope of $A_\lambda/E_{\mathrm{B-V}}$ vs. λ^{-1} should be 1.25 between the B- and V-band. Although the region between the B- and V-band is rather small, the interstellar extinction curve appears to be linear inside. This is indeed consistent with the expected almost linear slope of 1.25 that was found before.

Problem 4

a. The distribution of MRN gives the relative number of grains per interval in radius. It is given by

$$\frac{\mathrm{d}n}{\mathrm{d}a} \propto a^{-3.5}\,, \tag{17}$$

with upper and lower cutoffs at 0.25 μm and 0.005 μm, respectively. The average grain size can then by found by

$$\begin{aligned}
\langle a \rangle &= \frac{\int_{a_{\min}}^{a_{\max}} a\, n(a)\,\mathrm{d}a}{\int_{a_{\min}}^{a_{\max}} n(a)\,\mathrm{d}a} = \frac{\int_{a_{\min}}^{a_{\max}} a^{-2.5}\,\mathrm{d}a}{\int_{a_{\min}}^{a_{\max}} a^{-3.5}\,\mathrm{d}a} \\
&= \frac{\left[-\frac{2}{3}a^{-1.5}\right]_{a_{\min}}^{a_{\max}}}{\left[-\frac{2}{5}a^{-2.5}\right]_{a_{\min}}^{a_{\max}}} \\
&= \frac{5}{3}\left(\frac{0.25^{-1.5} - 0.005^{-1.5}}{0.25^{-2.5} - 0.005^{-2.5}}\right) = 8.31 \cdot 10^{-3}\ \mu\mathrm{m} \\
&= 1.66\, a_{\min}\,.
\end{aligned} \tag{18}$$

b. The same procedure as in question a can be used to find $\langle a^2 \rangle^{1/2}$:

$$\begin{aligned}
\langle a^2 \rangle &= \frac{\int_{a_{\min}}^{a_{\max}} a^2\, n(a)\,\mathrm{d}a}{\int_{a_{\min}}^{a_{\max}} n(a)\,\mathrm{d}a} = \frac{\int_{a_{\min}}^{a_{\max}} a^{-1.5}\,\mathrm{d}a}{\int_{a_{\min}}^{a_{\max}} a^{-3.5}\,\mathrm{d}a} \\
&= \frac{\left[-2a^{-0.5}\right]_{a_{\min}}^{a_{\max}}}{\left[-\frac{2}{5}a^{-2.5}\right]_{a_{\min}}^{a_{\max}}} \\
&= 5\left(\frac{0.25^{-0.5} - 0.005^{-0.5}}{0.25^{-2.5} - 0.005^{-2.5}}\right) = 1.07 \cdot 10^{-4}\ \mu\mathrm{m}^2\,.
\end{aligned} \tag{19}$$

Taking the square root of this gives us

$$\langle a^2 \rangle^{1/2} = 1.04 \cdot 10^{-2}\ \mu\mathrm{m}\,. \tag{20}$$

c. Again applying the procedure from question a gives the following for

$\langle a^3 \rangle^{1/3}$:

$$\begin{aligned}
\langle a^3 \rangle &= \frac{\int_{a_{\min}}^{a_{\max}} a^3\, n(a)\, \mathrm{d}a}{\int_{a_{\min}}^{a_{\max}} n(a)\, \mathrm{d}a} = \frac{\int_{a_{\min}}^{a_{\max}} a^{-0.5}\, \mathrm{d}a}{\int_{a_{\min}}^{a_{\max}} a^{-3.5}\, \mathrm{d}a} \\
&= \frac{\left[2a^{0.5}\right]_{a_{\min}}^{a_{\max}}}{\left[-\frac{2}{5}a^{-2.5}\right]_{a_{\min}}^{a_{\max}}} \\
&= -5\left(\frac{0.25^{0.5} - 0.005^{0.5}}{0.25^{-2.5} - 0.005^{-2.5}}\right) = 3.79 \cdot 10^{-6}\ \mu\mathrm{m}^3\,. \qquad (21)
\end{aligned}$$

Taking the cubic root of this gives us

$$\langle a^3 \rangle^{1/3} = 1.56 \cdot 10^{-2}\ \mu\mathrm{m}\,. \qquad (22)$$

Problem 5

a. The total volume of a GMC and one clump are given by using the diameter L that is given in Table 3.1 in the book. The diameter of a GMC is $L_{\mathrm{GMC}} = 50$ pc and of a clump is on average $L_{\mathrm{clump}} = 2$ pc.

$$\begin{aligned}
V_{\mathrm{GMC}} = &\ \tfrac{4\pi}{3}\left(\tfrac{L_{\mathrm{GMC}}}{2}\right)^2 &= 1.92 \cdot 10^{60}\,\mathrm{cm}^{-3} \qquad (23) \\
V_{\mathrm{clump}} = &\ \tfrac{4\pi}{3}\left(\tfrac{L_{\mathrm{clump}}}{2}\right)^2 &= 1.23 \cdot 10^{56}\,\mathrm{cm}^{-3}\,. \qquad (24)
\end{aligned}$$

Now we need to find how many clumps are present inside a typical GMC. For this we need to know how much of the total mass of the GMC is inside clumps. Turning to the last paragraph of section 3.1.2 in the book, we see that clumps comprise as much as 90% of the total molecular mass of $10^5\, M_\odot$. Thus

$$M_{\mathrm{clump}}^{\mathrm{total}} = 0.9 \cdot 10^5\, M_\odot = 9 \cdot 10^4\, M_\odot\,. \qquad (25)$$

Since according to this table $M_{\mathrm{clump}} = 30\, M_\odot$, there are

$$N_{\mathrm{clump}}^{\mathrm{total}} = \frac{9 \cdot 10^4}{30} = 3 \cdot 10^3\,, \qquad (26)$$

clumps inside a GMC. The volume of all the clumps combined is then

$$V_{\mathrm{clump}}^{\mathrm{total}} = V_{\mathrm{clump}}\, N_{\mathrm{clump}}^{\mathrm{total}} = 3.69 \cdot 10^{59}\,\mathrm{cm}^3\,. \qquad (27)$$

The filling factor is then given by

$$f = \frac{V_{\mathrm{clump}}^{\mathrm{total}}}{V_{\mathrm{GMC}}} = \frac{3.69 \cdot 10^{59}}{1.92 \cdot 10^{60}} = 0.192\,. \qquad (28)$$

b. The number of clumps per unit is mass is given by

$$\frac{\mathrm{d}N}{\mathrm{d}M} = N_0 \left(\frac{M}{M_{\mathrm{min}}}\right)^{-1.5}, \tag{29}$$

between $M_{\mathrm{min}} < M < M_{\mathrm{max}}$, where $M_{\mathrm{min}} = 30\,M_\odot$ and $M_{\mathrm{max}} = 10^3\,M_\odot$. To get the total number of clumps, the consant N_0 has to be found first. To do this we need to find the total mass of the clumps in the GMC. This mass can be found by integrating over $M\,\frac{\mathrm{d}N}{\mathrm{d}M}$ as such

$$M_{\mathrm{clump}}^{\mathrm{total}} = \int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} M\,N(M)\,\mathrm{d}M = \int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} M\,N_0 \left(\frac{M}{M_{\mathrm{min}}}\right)^{-1.5} \mathrm{d}M\,. \tag{30}$$

We already found in question a that $M_{\mathrm{clump}}^{\mathrm{total}} = 9 \cdot 10^4\,M_\odot$, thus formula 30 can be rewritten as

$$N_0 = \frac{M_{\mathrm{clump}}^{\mathrm{total}}}{\int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} M \left(\frac{M}{M_{\mathrm{min}}}\right)^{-1.5} \mathrm{d}M}\,. \tag{31}$$

The total number of clumps is found by integrating formula 29 and now formula 31 can be used to replace N_0

$$\begin{aligned}
N &= N_0 \int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} \left(\frac{M}{M_{\mathrm{min}}}\right)^{-1.5} \mathrm{d}M \\
&= 0.9 M_{\mathrm{clump}}^{\mathrm{total}} \frac{\int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} \left(\frac{M}{M_{\mathrm{min}}}\right)^{-1.5} \mathrm{d}M}{\int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} M \left(\frac{M}{M_{\mathrm{min}}}\right)^{-1.5} \mathrm{d}M} \\
&= 0.9 M_{\mathrm{clump}}^{\mathrm{total}} \frac{\int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} M^{-1.5}\,\mathrm{d}M}{\int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} M^{-0.5}\,\mathrm{d}M} \\
&= 0.9 M_{\mathrm{clump}}^{\mathrm{total}} \frac{\left[-2M^{-0.5}\right]_{M_{\mathrm{min}}}^{M_{\mathrm{max}}}}{\left[2M^{0.5}\right]_{M_{\mathrm{min}}}^{M_{\mathrm{max}}}} \\
&= 0.9 M_{\mathrm{clump}}^{\mathrm{total}} \frac{M_{\mathrm{min}}^{-0.5} - M_{\mathrm{max}}^{-0.5}}{M_{\mathrm{max}}^{0.5} - M_{\mathrm{min}}^{0.5}} \\
&= 520\,\mathrm{clumps}
\end{aligned} \tag{32}$$

However, if we assume that all the clumps have the same mean density as in question a this will lead to the same filling factor. The amount and sizes of the clumps may have changed, but the total mass is still $9 \cdot 10^4 M_\odot$ and the average density is also assumed to be the same and since $V = M/\rho$, the volume will remain the same.

c. The space between the clumps in a GMC is assumed to be filled by a HI solar composition gas. This remaining volume is

$$V_{\mathrm{HI}} = V_{\mathrm{GMC}} - V_{\mathrm{clump}}\,N_{\mathrm{clump}}^{\mathrm{total}} = 1.55 \cdot 10^{60}\,\mathrm{cm}^3\,, \tag{33}$$

where the values from question a were used. I will assume that the average density of the HI gas to be $n_{\rm HI} = 30\ {\rm cm}^{-3}$, which has been taken from section 2.2.1 on page 38 of the book. The density of the HI gas is then given by

$$\rho_{\rm HI} = m_H\, n_{\rm HI}\, \mu_{\rm HI} = 1.67 \cdot 10^{-24}\,{\rm g} \cdot 30\,{\rm cm}^{-3} \cdot 1.3 = 6.51 \cdot 10^{-23}\,{\rm g/cm}^{-3}\,. \tag{34}$$

The total mass of the HI gas is then

$$M_{\rm HI} = V_{\rm HI}\, \rho_{\rm HI} = 1.01 \cdot 10^{38}\,{\rm g} = 5.05 \cdot 10^4\, M_\odot\,. \tag{35}$$

This is about 50% of the total molecular mass of the GMC.

Problem 6

a. The clumps will have a mean free path given by $\lambda_{\rm mfp} = (n\sigma)^{-1}$, where n is the number density of the clumps and σ is the cross section of a clump. The cross section is assumed to be the projected area, thus taking from Table 3.1 in the book that the typical clump diameter is 2 pc, the cross section of a clump is

$$\sigma = \pi R^2 = 2.99 \cdot 10^{37}\,{\rm cm}^2\,. \tag{36}$$

From question *5a*, where typical clump properties were used, there are $\sim 3 \cdot 10^3$ clumps in a GMC. The GMC has a volume of $1.92 \cdot 10^{60}\,{\rm cm}^3$, thus

$$n = N/V = 1.56 \cdot 10^{-57}\,{\rm cm}^{-3}\,. \tag{37}$$

The mean free path of a clump is then

$$\lambda_{\rm mfp} = \frac{1}{n\sigma} = 2.14 \cdot 10^{19}\,{\rm cm}\,, \tag{38}$$

or 6.93 pc.
The relative speed of the clumps is assumed to be the virial speed given by $v_{\rm vir} = \sqrt{GM/R}$, where M and R correspond to the whole GMC. Taking the values of M and R from Table 3.1 gives us a virial speed of

$$v_{\rm vir} = \sqrt{\frac{6.673 \cdot 10^{-11} \cdot (10^5 \cdot 2 \cdot 10^{30})}{25 \cdot 3.086 \cdot 10^{16}}} = 4.16\,{\rm km/s}\,. \tag{39}$$

Thus the average collision time is given by dividing the mean free path by the virial speed

$$t_{\rm coll} = \lambda_{\rm mfp}/v_{\rm vir} = 1.63 \cdot 10^6\,{\rm yr}\,. \tag{40}$$

Note that this timescale is even shorter than the free fall timescale of $(G\rho)^{-1/2} \sim 5 \cdot 10^6\,{\rm yr}$. The assumptions made about the clumps are probably too simplistic and if we turn to the paper of Blitz & Shu (1980) we see that they get a value of $t_{\rm coll} \sim 10^7$ yr, which still is only slightly longer than their $t_{\rm ff}$ of $\sim 2 \cdot 10^6$ yr.

b. The effect of gravity can affect the motion of a clump before impact. The closest approach can be significantly less than the initial impact parameter b. The impact parameter is defined as the perpendicular distance between the path of a projectile and the center of the object that the projectile is approaching.
To get find an algebraic expression for $f_{\rm grav}$ we need to take energy conservation and angular momentum conservation into account. We look in the frame where one of the clumps, 'the projectile', is moving and the other clump that the projectile is approaching, 'the target', is standing still. The initial angular momentum is then given by

$$\begin{aligned} L_{\rm init} &= r \times p = r \times mv_0 \\ &= r \sin\theta \, mv_0 = bmv_0 \,, \end{aligned} \tag{41}$$

where θ is the angle between the velocity vector and radius towards the target, m is the mass of the projectile and v_0 is its initial (relative) velocity. In the last equation the fact that $r\sin\theta = b$ was used.
At the closest approach the angle between r and p is a right angle so that $L_{\rm end} = r \times p = Rmv_{\rm max}$, where R is the radius of the target and $v_{\rm max}$ is the speed of the projectile. Conservation of angular momentum then implies that

$$L_{\rm init} = L_{\rm end} \rightarrow bmv_0 = Rmv_{\rm max} \rightarrow v_{\rm max} = \frac{bv_0}{R} \,. \tag{42}$$

Conservation of energy at these times gives the following relation

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_{\rm max}^2 - \frac{GmM}{R} \,, \tag{43}$$

where M is the mass of the target. Dividing out m and replacing $v_{\rm max}$ by formula 42 implies that

$$v_0^2 = \left(\frac{bv_0}{R}\right)^2 - \frac{2GM}{R} \rightarrow b^2 = R^2\left(1 + \frac{2GM}{Rv_0^2}\right) \,, \tag{44}$$

The cross section of the target is thus effectively increased from πR^2 to $\pi R^2\left(1 + 2GM/Rv_0^2\right)$. The gravitational focusing factor is therefore

$$f_{\rm grav} = \left(1 + \frac{2GM}{Rv_0^2}\right) \tag{45}$$

Filling in the numbers for M, R and v_0 taken from question a or Table 3.1, $t_{\rm coll}$ is reduced by a factor of

$$f_{\rm grav}^{-1} = \left(1 + \frac{2 \cdot 6.673 \cdot 10^{-11} \cdot 30 \cdot 2 \cdot 10^{30}}{1 \cdot 3.086 \cdot 10^{16} \cdot (4.16 \cdot 10^3)^2}\right)^{-1} = 0.985 \tag{46}$$

It appears that graviational focusing is only of small importance for the clumps in the GMC.