Solutions to Star Formation Homework 2, Assigned by Steven Stahler

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Problem 1

Part a

We use the given production and destruction rates to write an equation for the rate of change of $n_{\rm OH}$ with time:

$$\frac{d(n_{\rm OH})}{dt} = p_1 \zeta n_{H_2} - \frac{n_{\rm OH}}{\tau}$$

When the system reaches a steady state, the left-hand side of this equation will be zero. We can then solve for the ratio of the number densities to obtain

$$\frac{n_{\rm OH}}{n_{\rm H_2}} = p_1 \zeta \tau$$

This ratio does not depend on the cloud density, as desired.

Part b

We can write another equation for the rate of change $n_{\rm OH}$ given the new destruction rate:

$$\frac{d(n_{\rm OH})}{dt} = p_1 \zeta n_{\rm H_2} - (k)(n_{\rm OH})(n_{\rm O})$$

But we are told that the ratio $n_{\rm O}/n_{\rm OH}$ is constant. If we denote this constant by α, then we may replace $n_{\rm O}$ in the above equation with $(\alpha)(n_{\rm H_2})$. The equation becomes:

$$\frac{d(n_{\rm OH})}{dt} = n_{\rm H_2}\left[p_1 \zeta - (k)(n_{\rm OH})(\alpha)\right]$$

Once again, when the system reaches a steady-state, the left-hand side goes to zero. Since n_{H_2} is not zero, the only way this condition can be achieved is if

$$p_1 \zeta = (k)(n_{\rm OH})(\alpha) \implies n_{\rm OH} = \frac{p_1 \zeta}{k\alpha}.$$

We see that in this case $n_{\rm OH}$ does not depend on cloud density, as claimed.

Part c

The angular excursion is related to the derivative of y with respect to x by

$$\tan(\Delta\theta) = \frac{\partial y}{\partial x}$$

If the wiggles are small, then we may make the approximation $\tan(\Delta\theta) \approx \Delta\theta$. This yields the first desired relation.

The rate of change of the apparent wiggle height with respect to time should be equal to the component of the turbulent velocity perpendicular to the field line, in the plane of the sky. Call this component v_x. We can write

$$\Delta V_{\text{turb}}^2 = v_x^2 + v_y^2 + v_z^2$$

Assuming the turbulence is isotropic, the three terms on the right-hand side should be of equal magnitude. This gives us

$$v_x^2 = \dot{y}^2 = \frac{1}{3}\Delta V_{\text{turb}}^2$$

and we have our second desired relation.

Part d

We use the wave solution for y as a function of x , as well as the results from part c above, to write

$$\left(\frac{\partial y}{\partial t}\right)^2 = k^2 V_A^2 y_0^2 \sin^2(kx - kV_A t) = \frac{1}{3}(\Delta V_{\text{turb}})^2$$

$$\left(\frac{\partial y}{\partial x}\right)^2 = k^2 y_0^2 \sin^2(kx - kV_A t) = (\Delta\theta)^2$$

We can now divide the top equation by the bottom equation, write V_A in terms of ρ and $B_\perp$, and solve for $B_\perp$ to obtain

$$B_\perp = \sqrt{\frac{4\pi\rho}{3}}\frac{\Delta V_{\text{turb}}}{\Delta\theta}$$

Part e

Plugging in the given values, and using $\rho = n_{\text{H}_2}(2m_{\text{H}})$ we find

$$B_\perp = \sqrt{\frac{(4/3)\pi(700)(2)(1.67 \times 10^{-24})(1.7 \times 10^5)^2}{(29\pi/180)^2}} \approx 33\ \mu G$$

Problem 2

Part a

First, we can establish that since deuterium has a more massive nucleus than hydrogen, its ionization potential will be greater, $I_{\text{D}} > I_{\text{H}}$. Then, if we define $\Delta E \equiv I_{\text{D}} - I_{\text{H}}$, we have $\Delta E > 0$. The rate constants will then be related via a Boltzmann factor in the following manner:

$$\frac{k_{\text{forward}}}{k_{\text{backward}}} = e^{-\frac{\Delta E}{k_{\text{B}} T}}$$

We conclude that the backward reaction is favored. To be more quantitative we need to compute ΔE. Let μ_{ed} denote the reduced mass of the electron when a deuteron is the nuclues, and let μ_{ep} denote the reduced mass of the electron when a proton is the nucleus. Then, using the formula for reduced mass, and the fact that the ionization potential is proportional to reduced mass, we find

$$\frac{I_{\rm D}}{I_{\rm H}} = \frac{\mu_{ed}}{\mu_{ep}} = \frac{1+m_e/m_p}{1+m_e/m_d} \approx 1 + \frac{m_e}{m_p} - \frac{m_e}{m_d} = 1 + \frac{m_e}{m_p}\Big(1 - \frac{m_p}{m_d}\Big)$$

Since $\frac{m_p}{m_d} \approx \frac{1}{2}$, we have

$$\frac{\mu_{ed}}{\mu_{ep}} \approx 1 + \frac{m_e}{2m_p}$$

This lets us derive a simple expression for ΔE:

$$\Delta E = I_{\rm D} - I_{\rm H} = I_{\rm H}\Big(\frac{I_{\rm D}}{I_{\rm H}} - 1\Big) = \frac{I_{\rm H}}{2}\frac{m_e}{m_p}$$

Plugging in the values of the physical constants and letting $T = 100{\rm K}$ we have $\frac{\Delta E}{k_{\rm B}T} = 0.43$ and so

$$\frac{k_{\rm forward}}{k_{\rm backward}} = e^{-0.43} = 0.65$$

Part b

We can write an equation for the rate of change of the deuterium concentration with respect to time in terms of the forwards and backwards rate constants.

$$\frac{d\big(n_{D^+}\big)}{dt} = (k_{\rm forward})(n_{H^+})(n_D) - (k_{\rm badkward})(n_H)(n_D^+)$$

When a steady state is reached the left-hand side equals zero. We can then solve for the density ratios in terms of the ratio of the rate constants, which we found in part a and which we will label as $\gamma \equiv 0.65$. We obtain

$$\frac{n_{D^+}}{n_D} = \gamma\frac{n_{H^+}}{n_H}$$

Part c

The rate of change of n_{HD} is given by

$$\frac{d\big(n_{HD}\big)}{dt} = (k_{HD})(n_{D^+})(n_{H_2}) - \frac{n_{HD}}{\tau_{\rm photo}}$$

Setting the left-hand side to zero for a steady sate, we solve for n_{HD} to obtain

$$n_{HD} = (k_{\rm HD})(n_{D^+})(n_{H_2})(\tau_{\rm photo})$$

Part d

The rate of change of n_{H^+} is given by

$$\frac{d\big(n_{H^+}\big)}{dt} = \zeta(n_H) - (k_{\rm OH})(n_{H^+})(n_O)$$

From part b, we know that

$$n_{H^+} = n_H\frac{n_{D^+}}{n_D}\frac{1}{\gamma}$$

Then we can use our result from part c to rewrite n_{D^+} in terms of the other constants. Finally, setting the rate of change of the hydrogen ion concentration to zero and solving for ζ we obtain

$$\zeta = \frac{k_{\rm OH}}{k_{\rm HD}}\frac{n_O}{n_H}\frac{n_H}{n_D}\frac{n_{\rm HD}}{n_{H_2}}\Big(\frac{1}{\tau_p\gamma}\Big)$$

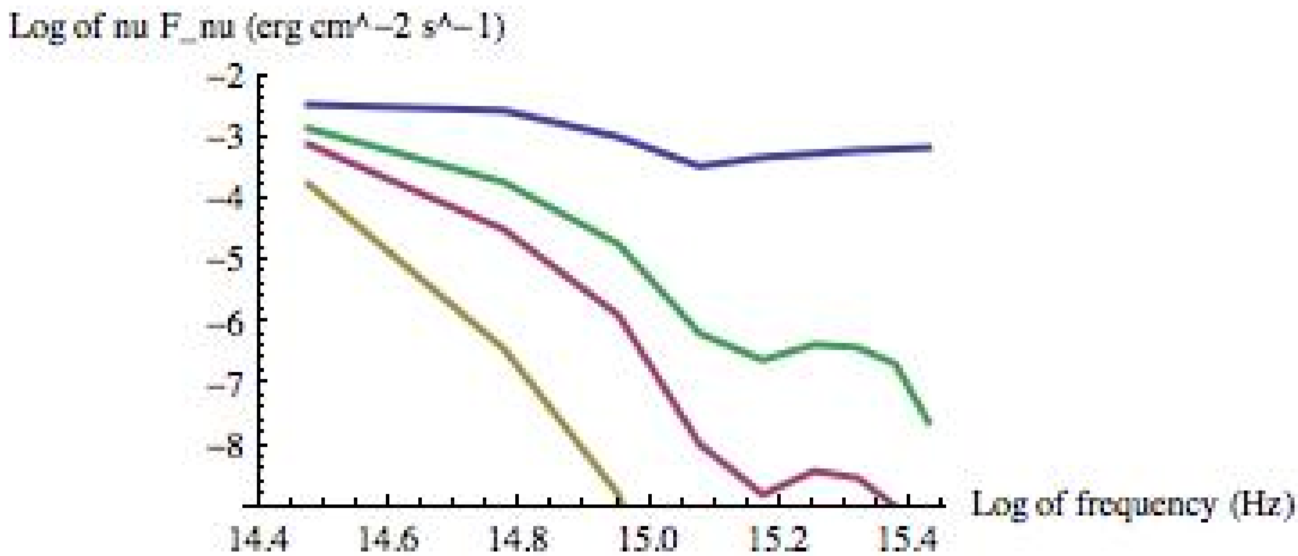

Figure 1: Blue curve is $A_V = 0$, Green curve is $A_V = 3$, pink curve is $A_V = 5$, goldl curve is $A_V = 10$

Problem 3

Part a

The energy density of a gas of point-like particles is given by

$$u = \frac{3}{2} nKT$$

We may then solve this equation for temperature and plug in the appropriate values of n and u to find that the temperature is approximately 85 Kelvin. This roughly agrees with the listed temperature of 80 K found in figure 2.2 of the text, and suggests that the cloud is in thermal equilibrium with the radiation field.

Part b

Since the radiation is isotropic, the specific intensity of the interstellar radiation, I_ν, equals the mean intensity J_ν. From the definition of radiative flux in terms of specific intensity, we find

$$\nu F_\nu = \nu \int_0^{2\pi} d\phi \int_0^{\pi/2} (J_\nu \cos\theta) \sin\theta d\theta = \pi \nu J_\nu$$

Part c

The graphs are shown in figure 1. To make these graphs, I calculated the value of $E(B-V)$ for each visual extinction by using the relation $E(B-V) = RA_V$, and then determined the extinction at each frequency using the visual extinction curve. I converted these extinctions into optical depths using equation 2.26, and then multiplied the values of νJ_ν from figure 7.4 by π times e raised to the negative optical depth. Note the small bump seen in the flux in the extreme UV regime when significant extinction is present.

Problem 4

Part a

In the first assignment we determined how an attractive force such as gravity can focus collisions so that the cross-section becomes larger than the geometric cross-section. In this problem we have focusing by Coulomb attraction, rather than gravity. An analysis identical to the one presented in the last assignment yields

$$\sigma = \pi a^2 \left(1 + \frac{2Ze^2}{m_e a v^2}\right)$$

Part b

We perform an average over the Boltzmann speed distribution:

$$n_e < v\sigma >= \int_0^\infty (n_e)(v)(\pi a^2)\left(1 + \frac{2Ze^2}{m_e a v^2}\right)\left(\frac{m_e}{2\pi k_b T}\right)^{3/2} 4\pi v^2 \exp\big(-\frac{mv^2}{2k_b T}\big) dv$$

We may then perform the change of variable

$$x \equiv \frac{mv^2}{2k_b T}$$

Our expression for the rate of incoming electrons becomes

$$\mathcal{R}_{\text{in}} = n_e \pi a^2 \sqrt{\frac{k_b T}{2m_e}} \frac{4\pi}{\pi^{3/2}} \Big[\int_0^\infty x e^{-x} dx + \frac{Ze^2}{ak_b T}\int_0^\infty e^{-x} dx\Big]$$

Both of the definite integrals evaluate to exactly unity. So we finally have

$$\mathcal{R}_{\text{in}} = n_e \pi a^2 \sqrt{\frac{8k_b T}{\pi m_e}} \Big[1 + \frac{Ze^2}{ak_b T}\Big]$$

We note that every factor in the above expression has a direct physical interpretation: $\mathcal{R}_{\text{in}}$ equals the number density of electrons times the geometric cross-section times the mean velocity of a Maxwellian distribution times a correction factor based on the ratio of electric potential energy at the surface of the grain over thermal energy.

Part c

As we showed in problem 3b, the radiative flux at a frequency ν impinging on a planar surface is πJ_ν (from an isotropic radiation field). Since the surface area is $4\pi a^2$, the energy per time hitting the surface is $4\pi^2 a^2 J_\nu$. Thus, the rate at which photons hit is $\frac{4\pi^2 a^2 J_\nu}{h\nu}$. We multiply by y to account for the photon conversion efficiency, and integrate over the suitable frequency range, to obtain the desired expression.

Part d

According to figure 7.4 in the text, the quantity νJ_ν is roughly constant at $10^{-3.9}$ erg cm^{-2} s^{-1} in the range of frequencies corresponding to energies between 10 and 13.6 eV, which are $\nu_1 = 2.418 \times 10^{15}$ Hz to $\nu_2 = 23.289 \times 10^{15}$ Hz. Therefore the integral in the expression for $\mathcal{R}_{\text{out}}$ can be simplifed to

$$\int_{\nu_1}^{\nu_2} \frac{\nu J_\nu}{\nu^2 h} d\nu \approx \frac{10^{-3.9} \text{ erg cm}^{-2} \text{ s}^{-1}}{h} \int_{\nu_1}^{\nu_2} \frac{1}{\nu^2} d\nu = \frac{10^{-3.9} \text{ erg cm}^{-2} \text{ s}^{-1}}{h} \Big(\frac{1}{\nu_1} - \frac{1}{\nu_2}\Big)$$

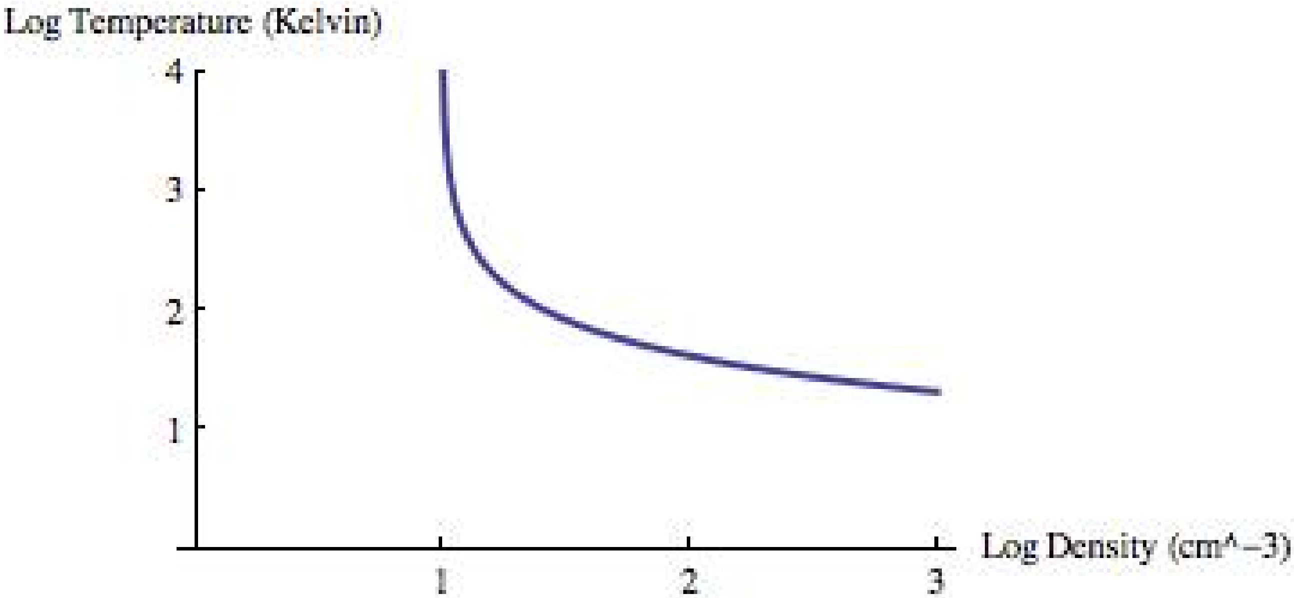

Figure 2: Approximate temperature versus density of interstellar gas

After plugging in the appropriate values for a, n_e and T and setting $\mathcal{R}_{\text{in}} = \mathcal{R}_{\text{out}}$, our equation reduces to approximately

$$(9.76^{-5}\text{ Hz})[1 + Z(1.67)] = 7.24 \times 10^{-4}\text{ Hz} \implies Z \approx 4$$

Problem 5

Parts a and b

An expression for the heating rate Γ_{PE} is found at the bottom of page 193, and an expression for the cooling rate Λ_{CII} is found near the bottom of page 200. Setting them equal and solving for temperature yields

$$T_g = \frac{92\text{K}}{\ln\left(\frac{(0.1)n_H}{\text{cm}^{-3}}\right)}$$

We can find an expression for P/k_b by multiplying the above expression for temperature by n_H. The plots can be found in figures 2 and 3.

Note that these plots match those found in page 39 of the textbook at high density, but they begin to vastly overestimate the temperature and pressure at densities below about 5 cm^{-3}, and eventually they diverge as density drops towards zero. This disagreement is to be expected, because we learn in section 2.2 of the text that at low densities, Lyman alpha emission becomes an important cooling mechanism, an effect that we have not accounted for in this analysis.

Part c

We must solve the following transcendental equation for n_H:

$$3000\text{ K cm}^{-3} = \frac{n_H(92\text{ K})}{\ln\left(\frac{0.1 n_H}{\text{cm}^{-3}}\right)}$$

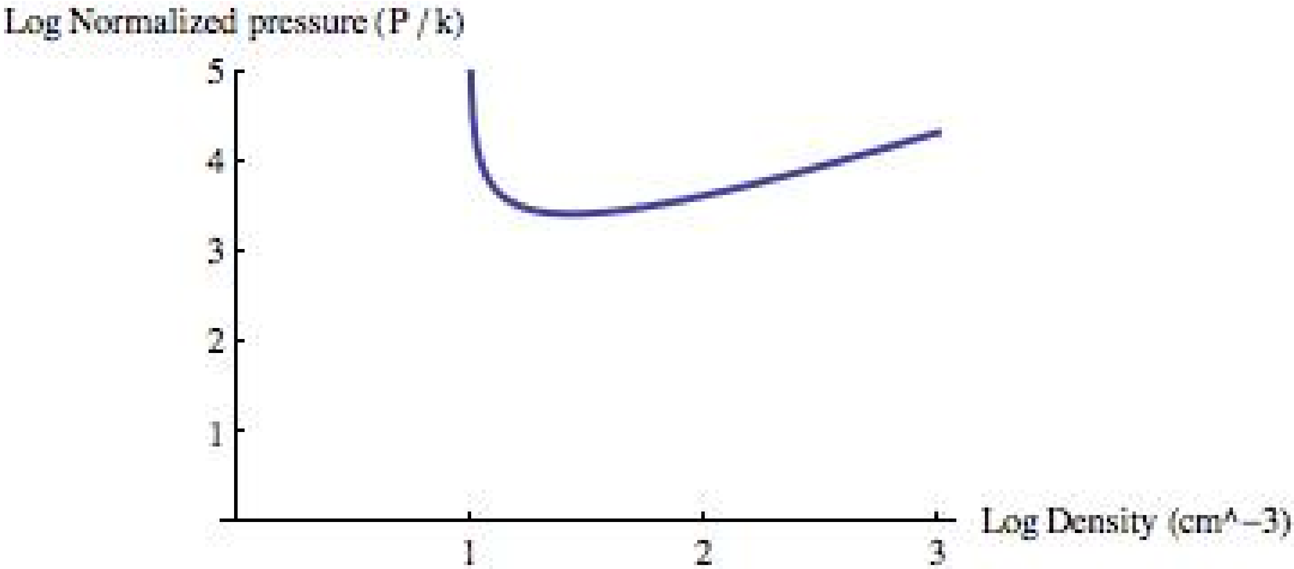

Figure 3: Approximate P/k_b versus density of interstellar gas

Note that the shape of the graph in figure 3 implies that we should expect only two values of n_H to satisfy this equation, rather than three as was the case in section 2.2 when a more exact cooling function was used. Therefore, we expect the smaller solution to correspond to point B in figure 2.5 from the text, rather than point A. We must keep this limitation in mind when we compute a density and temperature for the warm neutral medium.

Using a computer algebra system such as Mathematica, the two solutions to the above equation are approximately 17 and 56 cm^{-3}, which correspond to temperatures of approximately 180 K and 53 K. The latter temperature is a near perfect match to the value of 50 K given in the text. The former temperature is about a factor of 40 too small compared to the actual temperature of the warm neutral medium, which is given in the text as approximately 7000 K. This disagreement for the warm temperature is to be expected, given the limitations of our model discussed above.

Problem 6

Part a

Kirchoff's law tells us

$$B_\nu = \frac{j_\nu}{\rho\kappa_\nu}$$

In the statement of the problem we are also given an equation to express j_ν in terms of $\Lambda^*_{\rm CO}$ and Δ_ν. Then using the expression for F_{CO} given in the problem, and substituting for B_ν and j_ν using the aforementioned relations, the factors of Δ_ν cancel and we find

$$F_{\rm CO} = \frac{\Lambda^*_{\rm CO}}{4\rho\kappa_\nu}$$

Part b

We start with our expression for F_{CO} from part a and then substitute for $\rho\kappa_\nu$ using the relation given in the problem. We now have

$$F_{\mathrm{CO}} = \frac{\Lambda^*_{\mathrm{CO}} D}{4\tau_{J+1,J}}$$

Next we plug in for $\tau_{J+1,J}$ using equation 7.30, and for Λ^*_{CO} using the first equality of 7.35. Our expression for F_{CO} now contains a factor of τ_{10} in the denominator. We will use equation 7.28 to substitute for τ_{10}, but there are a few considerations we must keep in mind:

First, the $J = 1$ state is 3 times degenerate (the quantum number M can take values of -1, 0, or 1), while there is only one state corresponding to $J = 0$. Thus, the ratio $\frac{g_1}{g_0}$ that appears in equation 7.28 is equal to 3.

Second, we note that the partition function Q that appears in equation 7.28 is equal to $\frac{2k_bT}{h\nu_{10}}$, and using equation 7.31 this is equal to 2θ.

Third, the column density N_{CO} that appears in equation 7.28 can be rewritten as $n_{\mathrm{CO}}D$.

As a final note, the factor ΔE_{10} that appears in the expression for Λ^*_{CO} can be rewritten as $h\nu_{10}$.

After making the above substitutions and performing the immediate cancellations, we have

$$F_{\mathrm{CO}} = \frac{2\pi h\nu_{10}^4}{c^3}(J+1)^4\Delta V \frac{\exp\left[-\frac{(J+1)(J+2)}{2\theta}\right]\exp\left[\frac{J(J+1)}{2\theta}\right]}{1-\exp\left[-\frac{(J+1)}{\theta}\right]}$$

This is almost what we want, except for that final factor containing exponentials. We can add the two exponents in the numerator and the fraction becomes

$$\frac{\exp\left[-\frac{(J+1)}{\theta}\right]}{1-\exp\left[-\frac{(J+1)}{\theta}\right]}$$

But now if we multiply by one in the form of $\exp\left[\frac{(J+1)}{\theta}\right]/\exp\left[\frac{(J+1)}{\theta}\right]$, then our final expression for F_{CO} matches that of the first equality in 7.34.