AY250 Assignment 3

due: Thursday, Oct 7, 2010

1 - Consider an infinite planar slab of gas, whose density ρ only varies in the z-direction. The slab is isothermal, with a uniform sound speed a_T. The structure is in hydrostatic equilibrium, where the only source of gravity is the gas itself. We wish to find $\rho(z)$, where z is the height above the midplane.

(a) Write down the condition of hydrostatic equilibrium and Poisson's equation. Combine these to find a second-order differential equation for $\rho(z)$.

(b) Define a nondimensional density $\delta \equiv \rho/\rho_\circ$ and a nondimensional height $\zeta \equiv \sqrt{4\pi G\rho_\circ/a_T^2}\, z$. Using these new variables, write a simplified differential equation for $\delta(\zeta)$. What are the boundary values $\delta(0)$ and $\delta'(0)$?

(c) Solve this equation for $\delta(\zeta)$ analytically. Transform your answer into an expression for $\rho(z)$.

2 - Let us calculate the gravitational potential energy $\mathcal{W}$ of a pressure-bounded, singular isothermal sphere. That is, we will determine the coefficient f in the expression

$$\mathcal{W} = -f\,\frac{G\,M_\circ^2}{R_\circ}\ ,$$

for a sphere of mass $M_\circ$ and radius $R_\circ$. We will do the calculation two ways. (a) Calculate U, the sphere's thermal energy. Express your result in terms of G, $M_\circ$, and $R_\circ$.

(b) From Appendix D, the virial theorem in this case reads

$$2\,U + \mathcal{W} = \mathcal{P}$$

where

$$\mathcal{P} \equiv \int P\,\boldsymbol{r}\cdot\boldsymbol{n}\,d^2\boldsymbol{x}\ .$$

Using your result from part (a), calculate $\mathcal{W}$, in terms of the same three quantities. Thus, determine f.

(c) As a complementary approach, first use Poisson's equation (9.5a) to calculate the gravitational potential Φ_g. For the density on the righthand side, use equation (9.8). This will give you Φ_g *inside* the sphere, up to an additive constant. Choose this constant so that $\Phi_g(r)$ joins smoothly to the proper external potential, *i.e.*, one that goes to zero at infinite distance.

(d) Finally, use the general expression

$$\mathcal{W} = \frac{1}{2} \int \rho \, \Phi_g \, d^3 \boldsymbol{x} \; ,$$

to derive the second form for the potential energy. Verify that f agrees with your previous answer.

3 - We may understand the gravitational stability of a magnetized cloud by application of the virial theorem. The analysis is approximate, but physically instructive.

(a) Suppose the cloud is an isothermal sphere of mass M, radius R, and sound speed a_T. Use the virial theorem in the form of equation (3.16) (with $\mathcal{T} = 0$) to find an expression for B, the mean magnetic field. Ignore factors of order unity when evaluating the gravitational energy $\mathcal{W}$.

(b) Suppose now that the cloud is so cold that we may neglect its thermal energy entirely. Find the cloud's mass, M_Φ, in terms of G and Φ_{cl}, the magnetic flux that threads it. Compare your answer to equation (9.58).

(c) In the case where U is *not* negligible, the critical mass M_{crit} is larger than M_Φ. Again applying the virial theorem, together with your result from (b), find M_{crit} in terms of M_Φ and the M_J, the Jeans mass in equation (9.24). Your result will be an *implicit* equation for M_{crit}.

(d) If we now equate M_J and M_{BE}, then your last result gives M_{crit}, again implicitly, in terms of M_{BE} and M_Φ. By contrast, equation (9.57) is an *explicit* solution. Show numerically, by testing various values of M_{BE}/M_Φ, that the two equations give similar values of M_{crit}/M_Φ. Indicate the limit of validity of equation (9.57) as an approximate solution. That is, how large can M_{crit} be relative to M_ϕ for the equation to hold?

4 - Let us explore further Alfvén waves traveling in some oblique direction to the background magnetic field $\boldsymbol{B}_\circ$. Consider first their *phase* velocity V_{phase}, i.e, the speed at which the field disturbance travels. From the dispersion relation of equation (9.81), $V_{\mathrm{phase}} \equiv \omega/k = V_A \cos\theta_B$, where θ_B is the angle between $\boldsymbol{B}_\circ$ and the wave vector $\boldsymbol{k}$.

(a) Next determine the wave's *group* velocity, which characterizes its transmission of energy. In general, the vector group velocity $\boldsymbol{V}_{\mathrm{group}}$ is found from the dispersion relation $\omega = \omega(\boldsymbol{k})$ by

$$\boldsymbol{V}_{\mathrm{group}} \equiv \frac{\partial \omega}{\partial \boldsymbol{k}} \; .$$

Find the direction and magnitude of $\boldsymbol{V}_{\mathrm{group}}$ for oblique Alfvén waves.

(b) Show that δu is still antiparallel to δB, just as in Alfvén waves that travel along the background field. Find the scalar coefficient relating these two vectors and compare your answer to equation (9.89).

(c) From (b), find the ratio of the magnetic energy density of the wave, $|\delta B|^2/8\pi$, to its kinetic energy density $(\rho_\circ/2)\,|\delta u|^2$.

(d) The wave is associated with a current density

$$\delta \boldsymbol{j} \;=\; \frac{c}{4\,\pi}\,\boldsymbol{\nabla} \times \delta \boldsymbol{B}\;,$$

where $\delta \boldsymbol{B}\,(\boldsymbol{r},t)$ now represents the full, traveling wave perturbation, as in equation (9.61). This current, in turn, creates a Lorentz force, whose magnitude per unit volume is

$$f \;=\; \frac{\delta \boldsymbol{j}}{c}\, \times \, \boldsymbol{B}_\circ\;.$$

Show that the force driving the wave is entirely due to magnetic tension rather than magnetic pressure.

5 - The physics of ambipolar diffusion is elucidated by considering an infinite, planar slab of self-gravitating gas, initially supported by a combination of thermal and magnetic pressure. The gas is isothermal, with sound speed a_T, and the magnetic field $\boldsymbol{B} = B\,\hat{x}$ lies wholly in the plane. Any initial field gradient $\partial B/\partial z$ smooths out with time. Here, z is the coordinate normal to the plane. Our goal here is to show that $B(z,t)$ is governed by a diffusion equation.

(a) Consider the ion-neutral drift velocity $\boldsymbol{v}_{\mathrm{drift}}$. Rewrite equation (10.3) in terms of the z-gradient of the magnetic pressure, $B^2/(8\,\pi)$. Your expression should also contain the neutral and ion mass densities, ρ and ρ_i, respectively, as well as the ion-neutral drag coefficient $\gamma \equiv \langle \sigma_{\mathrm{in}} u_i' \rangle/m_i$, where m_i is the ion mass. In what direction is $\boldsymbol{v}_{\mathrm{drift}}$? Interpret your answer physically.

(b) Show that equation (10.4) for the evolution of $\boldsymbol{B}$ can be rewritten as an expression for the convective time derivative of the scalar quantity B/ρ. The convective time derivative is

$$\frac{D}{Dt} \;\equiv\; \left(\frac{\partial}{\partial t}\right)_z + u\left(\frac{\partial}{\partial z}\right)_t\;,$$

where u is the velocity of the neutrals.

(c) It is convenient to change the independent spatial variable from z to σ, the surface density from the midplane $z = 0$ to height z:

$$\sigma \;\equiv\; \int_0^z \rho\,(z',t)\,dz'\;.$$

Recast your expression for $\boldsymbol{v}_{\mathrm{drift}}$ from (a) in terms of the σ-gradient of magnetic pressure.

(d) Similarly, rewrite your result from (b) as a time-derivative of B/ρ at fixed σ. (*Hint*: You will need to invoke mass continuity, equation (3.7).) Combine these results to obtain a partial differential equation for $B(\sigma, t)$. Your equation will contain ρ_i, which is given in terms of ρ by equation (8.36).

(e) This final result should be in the form of a nonlinear diffusion equation. What is the characteristic time t_{diff} for a magnetic field of average magnitude B to diffuse out of a slab with total mass density σ_{tot}? Show that t_{diff} is a dimensionless multiple of $a_T/(G\,\sigma_{\mathrm{tot}})$, the vertical crossing time of the slab. (*Hint*: What is the relationship of B and σ_{tot} when thermal and magnetic pressures are comparable?)

6 - Equation (10.34) gives the density profile inside the rarefaction wave of a collapsing, protostellar cloud. This equation was derived assuming steady-state flow and neglecting the gravitational pull of material outside the protostar on the infalling gas. Let us test both of these assumptions.

(a) Let $R(t)$ be the radius of the rarefaction wave a time t after protostar formation. If the velocity is truly free-fall inside the wave, what is Δt, the transit time from $R(t)$ to the origin?

(b) What is $\Delta t/t$? Under what circumstances is the steady-state assumption valid?

(c) Using $\rho(r)$ from equation (10.34), what is the mass ΔM inside the wave?

(d) What is $\Delta M/M_*$? Can the gravity of the falling gas indeed be neglected?