AY250 Assignment 3: Solutions

Problem 1

(a) Hydrostatic equilibrium: $-1/\rho\nabla P - \nabla\Phi_g = 0$. Poisson's equation: $\nabla^2\Phi_g = 4\pi G\rho$. Also, we know that for isothermal material $P = \rho {a_T}^2$, meaning we can write:

$$\nabla^2\Phi_g = -\nabla\left(\frac{a_T^2}{\rho}\nabla\rho\right) = 4\pi G\rho. \tag{15}$$

Then, because we're dealing with an infinite plane of gas we can rewrite ∇ as $\partial/\partial z$. We get a second order differential equation for ρ:

$$\frac{a_T^2}{\rho^2}\left(\frac{\partial\rho}{\partial z}\right)^2 - \frac{a_T^2}{\rho}\frac{\partial^2\rho}{\partial z^2} = 4\pi G\rho. \tag{16}$$

(b) Using the dimensionless variables given, $\delta \equiv \rho/\rho_0$ and $\zeta \equiv \sqrt{4\pi G\rho_0/a_T^2}z$, and that

$$\frac{\partial}{\partial z} = \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta} \tag{17}$$

$$= \sqrt{4\pi G\rho_0/a_T^2}\,\frac{\partial}{\partial\zeta}, \tag{18}$$

we can rewrite the above differential equation as

$$\frac{1}{\delta^2}\left(\frac{\partial\delta}{\partial\zeta}\right)^2 - \frac{1}{\delta}\frac{\partial^2\delta}{\partial\zeta^2} - \delta = 0. \tag{19}$$

The density of this slab at $z = 0$ is ρ_0 therefore $\delta(\zeta = 0) = 1$. In other to get the other boundary equation we can integrate $d^2\phi_g/dz^2$ from $z = -\epsilon$ to $z = \epsilon$ from which we get

$$\frac{d\phi}{dz}\Big|_{-\epsilon}^{+\epsilon} = 4\pi G\int_{-\epsilon}^{+\epsilon}\rho dz. \tag{20}$$

Since $\rho(0) =$ finite, then $d\phi_g/dz(0) = 0$. From this we see that $\rho'(0) = 0$ and so $\delta'(0) = 0$, which gives our second boundary equation to solve this differential equation.

(c) We can solve the above equation. First, we rewrite the equation in terms of $\ln\delta$:

$$\frac{d^2\ln\delta}{d\zeta^2} = -\delta. \tag{21}$$

Then we multiply both sides by $d\ln\delta/d\zeta$ to get

$$\frac{d\ln\delta}{d\zeta}\frac{d^2\ln\delta}{d\zeta^2} = -\delta\frac{d\ln\delta}{d\zeta} = -\frac{d\delta}{d\zeta}, \tag{22}$$

which can be rewritten

$$\frac{1}{2}\frac{d}{d\zeta}\left(\frac{d\ln\delta}{d\zeta}\right)^2 = -\frac{d\delta}{d\zeta}. \tag{23}$$

We can integrate the above equation to get

$$\frac{1}{\delta^2}\left(\frac{d\delta}{d\zeta}\right)^2 = 2(1-\delta), \tag{24}$$

where we have chosen our constant of integration in order to meet the boundary conditions. If we then rearrange and integrate the above equation we find

$$\int_1^\delta \frac{d\delta}{\delta\sqrt{1-\delta}} = \sqrt{2}\zeta. \tag{25}$$

If we try $\delta = \text{sech}^2\theta$, find that the above equation becomes simply

$$\theta = \zeta/\sqrt{2}. \tag{26}$$

Thus, the solution to our differential equation can be written as

$$\delta = \text{sech}^2(\zeta/\sqrt{2}). \tag{27}$$

Transforming this equation back in terms of ρ we find

$$\rho(z) = \rho_0\,\text{sech}^2\left(\sqrt{\frac{2\pi G\rho_0}{a_T^2}}\;z\right). \tag{28}$$

Problem 2

(a) We know that $U = 3/2 \int P d^3x$ and that for a SIS $P = \rho a_T^2$ therefore

$$U = \frac{3}{2} \int \rho a_T^2 d^3x. \tag{29}$$

However, a_T does not vary spatially so we can take that outside the integral and $\int \rho d^3x = M_0$ so that we now have the following expression for U

$$U = \frac{3}{2} M_0 a_T^2. \tag{30}$$

Integrate over the density to find the mass (Eqn. 9.9 from the book), and using the fact that $\rho = a_T^2/2\pi G r^2$ we find that $a_T^2 = GM_0/2R_0$, which we can then substitute into our expression for U above

$$U = \frac{3}{4}\frac{GM_0^2}{R_0}. \tag{31}$$

(b) To calculate f we must use the virial theorem, which in this case reads as $2U + \mathcal{W} = \mathcal{P}$. We already know U so we are left to solve for $\mathcal{P}$, which is given as $\mathcal{P} \equiv \int P\vec{r} \cdot \vec{n} d^2x$. Thus,

$$\mathcal{P} = 4\pi R_0^3 P. \tag{32}$$

The pressure is being evaluated on the surface of the sphere, and using our previous equation to relate the pressure to the density and evaluating at R_0 we find $P(R_0) = a_T^4/2\pi G R_0^2$ and substituting into the equation above we get

$$\mathcal{P} = \frac{4\pi R_0^3 a_T^4}{2\pi G R_0^2} = \frac{GM_0^2}{R_0}. \tag{33}$$

Using this result, and our previous expression for U we find that

$$\mathcal{W} = -\frac{GM_0^2}{R_0}, \tag{34}$$

and hence $f = 1$.

(c) Using equation (9.5a) and equation (9.8) we find

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi_g}{dr}\right) = 4\pi G\rho, \tag{35}$$

and after substituting $\rho(r) = a_T^2/4\pi G r^2$ and canceling the appropriate terms ~~and differentiating~~ we get

$$\frac{d}{dr}\left(r^2 \frac{d\Phi_g}{dr}\right) = 2a_T^2. \tag{36}$$

Multiplying each side of this equation by dr and integrating we find

$$r^2 \frac{d\Phi_g}{dr} = 2a_T^2 r, \tag{37}$$

and if we separate variables and integrate we find

$$\Phi_g(r) = 2a_T^2 \ln r + C, \tag{38}$$

where C is a constant of integration. We want the potential to be smooth over the boundary, and outside the sphere it should look like $-GM_0/r$. Therefore, we select C such that $\Phi_g R_0 = -GM_0/R_0$ which yields

$$\Phi_g(r) = 2a_T^2 \ln(r/R_0) - GM_0/R_0. \tag{39}$$

(d) Using the given equation for W,

$$\begin{aligned} \mathcal{W} &= \frac{1}{2}\int \rho \Phi_g d^3x, & (40)\\ &= \frac{1}{2}\int_0^{R_0} \frac{a_T^2}{2\pi G r^2}[2a_T^2 \ln(r/R_0) - GM_0/R_0] 4\pi r^2 dr, & (41)\\ &= \frac{2a_T^4}{G}\int_0^{R_0} \ln(r/R_0) dr \;-\; M_0 a_T^2. & (42) \end{aligned}$$

From here we see that the integral can be solved if we substitute $x = r/R_0$, in which case we are left with

$$\mathcal{W} = -\frac{2a_T^4 R_0}{G} \;-\; M_0 a_T^2, \tag{43}$$

and using $a_T^2 = GM_0/2R_0$ we find

$$\mathcal{W} = -\frac{GM_0^2}{R_0}. \tag{44}$$

Therefore, as we already saw in part (b) $f = 1$.

Problem 3

a)

From (3.16) $2U + W + \mathcal{M} = 0$. Using, $U = \frac{3}{2}Ma_T^2$, $W = -\frac{GM^2}{R}$ and $\mathcal{M} = \frac{B^2}{8\pi}\frac{4}{3}\pi R^3$, we have,

$$\frac{B^2R^3}{6} = \frac{GM^2}{R} - 3Ma_T^2$$
$$\Rightarrow B = \left(\frac{6GM^2}{R^4} - \frac{18Ma_T^2}{R^3}\right)^{\frac{1}{2}}$$

b)

Taking $U = 0$,

$$\frac{B^2R^3}{6} = \frac{GM_\Phi^2}{R}$$
$$\Rightarrow M_\Phi = \left(\frac{B^2R^4}{6G}\right)^{\frac{1}{2}} = \frac{\Phi_{cl}}{\pi}\frac{1}{\sqrt{6G}},$$

using, $\Phi_{cl} = \pi R^2 B$ for the magnetic flux. Evaluating the constants,

$$M_\Phi = 0.13\frac{\Phi_{cl}}{\sqrt{G}}$$

This is very close to equation (9.58), which gives a prefactor of 0.12.

c)

Going back to the virial thereom:

$$\frac{B^2R^3}{6} = \frac{GM^2}{R} - 3Ma_T^2$$

Substituting in $M = \frac{4}{3}\pi R^3\rho$ in the third term and multiplying through by R/G,

$$\frac{B^2R^4}{6G} = M^2 - \frac{4\pi}{G}R^4\rho a_T^2$$

Using our equation for M_Φ from part b) and factoring out the third term,

$$M_\Phi^2 = M^2 - 4\pi R^4\rho^{4/3}\left(\frac{a_T^3}{\rho^{1/2}G^{3/2}}\right)^{2/3}$$

Substituting in, $M_J = \frac{a_T^3}{\rho^{1/2}G^{3/2}}$,

$$M_\Phi^2 = M^2 - 4\pi\left(\rho R^3\right)^{4/3}M_J^{2/3}$$

which becomes,

$$M_\Phi^2 = M^2 - 4\pi\left(\frac{3}{4\pi}\right)^{4/3}M^{4/3}M_J^{2/3}$$

The numeral factor is close to unity, so we have approximately

$$M_\Phi^2 = M^2 - M^{4/3}M_J^{2/3}$$

Dividing through by M^2,

$$\left(\frac{M_\Phi}{M}\right)^2 = 1 - \left(\frac{M_J}{M}\right)^{2/3}$$

We now identify M as the critical mass M_{crit}. We find

$$\frac{M_J}{M_{crit}} = \left[1 - \left(\frac{M_\Phi}{M_{crit}}\right)^2\right]^{3/2}$$

$$\Rightarrow M_{crit} = M_J\left[1 - \left(\frac{M_\Phi}{M_{crit}}\right)^2\right]^{-3/2}$$

d)

If $M_{crit} \approx M_\Phi + M_{BE}$, then $\frac{M_{crit}}{M_\Phi} \approx 1 + \frac{M_{BE}}{M_\Phi}$. Dividing our answer from part c) by M_Φ and equating M_J and M_{BE},

$$\frac{M_{crit}}{M_\Phi} = \frac{M_{BE}}{M_\Phi}\left[1 - \left(\frac{M_{crit}}{M_\Phi}\right)^{-2}\right]^{-3/2}$$

If we try out a value for $\frac{M_{BE}}{M_\Phi}$, the corresponding value of $\frac{M_{crit}}{M_\Phi}$ (from the above equation) should be approximately equal to $1 + \frac{M_{BE}}{M_\Phi}$ in order for equation 9.57 to hold. At some point, this relation will break down.

M_{BE}/M_Φ	M_{crit}/M_Φ	$1 + M_{BE}/M_\Phi$
0.05	1.07	1.05
0.10	1.12	1.10
1.0	1.78	2.0
5.0	5.28	6.0
8.0	8.18	9.0
10.0	10.16	11

By the time we get to $M_{BE}/M_\Phi = 10$, the difference between M_{crit}/M_Φ and $1 + M_{BE}/M_\Phi$ is about 8%.

Problem 4

a)

If we set the direction of B_0 as $\hat{z}$, then

$$k_z = kcos(\theta_B)$$

The dispersion relation becomes

$$\omega = k_z V_A$$

Thus

$$[V_{group}]_z = \frac{\partial \omega}{\partial k_z} = V_A$$
$$\text{while}\, [V_{group}]_x = [V_{group}]_y = 0$$

Thus energy travels along the magnetic field at speed V_A.

b)

Begin with equation (9.66):

$$-\omega \delta \vec{B} = (\vec{k} \cdot \vec{B_0})\delta \vec{u} - (\vec{k} \cdot \delta \vec{u})\vec{B_0}$$

For a transverse wave, $\vec{k} \cdot \delta \vec{u} = 0$. Setting $\vec{k} \cdot \vec{B_0} = k_z B_0$,

$$-\omega \delta \vec{B} = k_z B_0 \delta \vec{u}$$
$$\Rightarrow \delta \vec{u} = \frac{-\omega}{k_z B_0} \delta \vec{B}$$

and $\delta \vec{u}$ and $\delta \vec{B}$ are antiparallel. Using $\omega = k_z V_A$, we have

$$\frac{\delta \vec{u}}{V_A} = -\frac{\delta \vec{B}}{B_0}$$

which reproduces equation (9.89).

c)

$$
\begin{aligned}
\frac{E_{mag}}{E_{kin}} &= \frac{|\delta B|^2}{8\pi} \frac{2}{|\delta u|^2 \rho_0} \\
&= \frac{\delta B^2}{8\pi} \frac{2B_0^2}{\delta B^2 V_A^2 \rho_0} \\
&= \frac{B_0^2}{4\pi\rho_0 V_A^2} \\
&= 1
\end{aligned}
$$

from the definition of V_A.

d)

Plugging δj into f,

$$
\begin{aligned}
f &= \frac{\delta j \times B_0}{c} \\
&= \frac{1}{c}\left[\frac{c}{4\pi}(\nabla \times \delta B) \times B_0\right] \\
&= \frac{-1}{4\pi}\left[\nabla(B_0 \cdot \delta B) - (B_0 \cdot \nabla)\delta B\right]
\end{aligned}
$$

where we have expanded the triple cross product and used $\vec{\nabla} \cdot \vec{B_0} = 0$. From equation (9.80), $\delta u \cdot \vec{k} = 0$ implies that $\delta u \cdot B = 0$. Thus, from part b) $\delta B \cdot B_0 = 0$, and

$$
f = \frac{(B_0 \cdot \nabla)\delta B}{4\pi}
$$

which is the form of magnetic tension from equation (3.6).

Problem 5

a)

Starting with equation (10.3):

$$v_{drift} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi n n_e [m_n < \sigma_{in} u_{i'} >]}$$

To calculate the triple cross-product, we note that B is in the x-direction, and there are no x- or y-gradients. Thus,

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{\nabla B^2}{2}(-\hat{z})$$

$$\Rightarrow v_{drift} = -\frac{\nabla B^2}{8\pi}\frac{1}{\rho\rho_i\gamma}\hat{z}$$

where we have used $nm_n = \rho$. Our slab is partially supported against self-gravity by magnetic pressure. Hence, $\frac{B^2}{8\pi}$ decreases away from the midplane.

Thus v_{drift} is in the $+\hat{z}$ direction. Since $v_{drift} = u_i - u$, the neutrals are moving toward the midplane, in the $-\hat{z}$ direction.

b)

First taking the convective derivative of B/ρ and plugging in continuity:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial z}$$

$$\frac{D(B/\rho)}{Dt} = \frac{1}{\rho}\frac{\partial B}{\partial t} - \frac{B}{\rho^2}\left(-\rho\frac{\partial u}{\partial z} - u\frac{\partial \rho}{\partial z}\right) + \frac{u}{\rho}\frac{\partial B}{\partial z} - \frac{uB}{\rho^2}\frac{\partial \rho}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial B}{\partial t} + \frac{B}{\rho}\frac{\partial u}{\partial z} + \frac{u}{\rho}\frac{\partial B}{\partial z}$$

Equation (10.4) reads:

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \nabla \times (v_{drift} \times B)$$

Expanding the triple cross producut, and using $\vec{\nabla} \cdot \vec{B} = 0$,

$$\frac{1}{\rho}\frac{\partial B}{\partial t} = -\frac{u}{\rho}\frac{\partial B}{\partial z} - \frac{B}{\rho}\frac{\partial u}{\partial z} - \frac{v_{drift}}{\rho}\frac{\partial B}{\partial z} - \frac{B}{\rho}\frac{\partial v_{drift}}{\partial z}$$

Using the previous result

$$\frac{D(B/\rho)}{Dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}(B v_{drift})$$

c)

From the definition of σ,

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{B^2}{8\pi}\right) = \frac{\partial}{\partial \sigma}\left(\frac{B^2}{8\pi}\right)$$

Plugging this back into v_{drift} from a),

$$v_{drift} = -\frac{1}{\gamma \rho_i}\frac{\partial}{\partial \sigma}\left(\frac{B^2}{8\pi}\right).$$

d)

First we need to convert from derviatives in terms of z to σ. Using continuity:

$$\left(\frac{\partial \sigma}{\partial t}\right)_z = \int \frac{\partial \rho}{\partial t} dz = -\int \frac{\partial(\rho u)}{\partial z} dz = -\rho u$$

Since $u(0) = 0$. Next, find $\left(\frac{\partial}{\partial t}\right)_z$ in terms of $\left(\frac{\partial}{\partial t}\right)_\sigma$. Writing out the derivative explicitly,

$$\left(\frac{\partial}{\partial t}\right)_\sigma dt + \left(\frac{\partial}{\partial \sigma}\right)_t d\sigma = \left(\frac{\partial}{\partial z}\right)_t dz + \left(\frac{\partial}{\partial t}\right)_z dt$$

Holding z constant:

$$\left(\frac{\partial}{\partial t}\right)_\sigma = \left(\frac{\partial}{\partial t}\right)_z - \left(\frac{\partial}{\partial \sigma}\right)_t \left(\frac{\partial \sigma}{\partial t}\right)$$

$$\Rightarrow \left(\frac{\partial}{\partial t}\right)_z = \left(\frac{\partial}{\partial t}\right)_\sigma - (\rho u)\left(\frac{\partial}{\partial \sigma}\right)_t$$

Finally, we need to find $\left(\frac{\partial}{\partial z}\right)_t$ in terms of $\left(\frac{\partial}{\partial \sigma}\right)_t$

$$\left(\frac{\partial}{\partial z}\right)_t = \left(\frac{\partial}{\partial \sigma}\right)_t \left(\frac{\partial \sigma}{\partial z}\right) = \rho \left(\frac{\partial}{\partial \sigma}\right)_t$$

We can now write out the convective derivative in terms of σ,

$$\begin{aligned}\frac{D(B/\rho)}{Dt} &= \left(\frac{\partial(B/\rho)}{\partial t}\right)_z + u\left(\frac{\partial(B/\rho)}{\partial z}\right)_t \\ &= \left(\frac{\partial(B/\rho)}{\partial t}\right)_\sigma + \left(\frac{\partial(B/\rho)}{\partial \sigma}\right)_t (-\rho u) + \left(\frac{\partial(B/\rho)}{\partial \sigma}\right)_t \rho u \\ &= \left(\frac{\partial(B/\rho)}{\partial t}\right)_\sigma\end{aligned}$$

The convective derivative is just the time derivative with respect to σ. Putting this together with our result from parts b) and c),

$$\left(\frac{\partial(B/\rho)}{\partial t}\right)_\sigma = -\frac{1}{\rho}\left[\frac{\partial}{\partial z}(Bv_{dr})\right]_t = -\left[\frac{\partial}{\partial \sigma}(Bv_{dr})\right]_t$$

$$\Rightarrow \left(\frac{\partial(B/\rho)}{\partial t}\right)_\sigma = \frac{1}{\gamma}\frac{\partial}{\partial \sigma}\left[\frac{B^2}{4\pi\rho_i}\frac{\partial B}{\partial \sigma}\right]$$

e)

We can estimate the characteristic diffusion time using our answer from part d):

$$\frac{B/\rho}{t} \sim \frac{B^3}{\gamma \rho_i \sigma^2}$$
$$\Rightarrow t_{diff} \sim \frac{\gamma \sigma^2 \rho_i}{B^2 \rho}$$

From equation (8.36), $\rho_i = C\rho^{1/2}$. Using this,

$$\Rightarrow t_{diff} \sim \frac{\gamma \sigma^2 C}{B^2 \rho^{1/2}}$$

We can further reduce our expression for t_{diff} by first noting that, for hydrostatic balance,

$$g \sim \frac{GM}{R^2} \sim G\sigma$$

Thus, the thermal pressure is

$$P \sim \rho g h \sim \rho G \sigma h \sim G\sigma^2$$

Since $P = \rho a_T^2$, then approximate equality of thermal and magnetic pressure gives

$$B^2 \sim G\sigma^2 \text{ and } \rho a_T^2 \sim G\sigma^2$$

Substituting these relations into our expreesion for t_{diff},

$$t_{diff} \sim \frac{\gamma \sigma^2 C a_T}{G\sigma^2 G^{1/2} \sigma}$$
$$\sim \frac{\gamma C}{G^{1/2}} \frac{a_T}{G\sigma}$$

Thus,

$$t_{diff} \approx n \frac{a_T}{G\sigma},$$

where $n \equiv \frac{\gamma c}{G^{1/2}}$ is a dimensionless number.

Problem 6

(a) If we imagine that a parcel of gas is falling in from $R(t)$ at the freefall velocity $v_{\rm ff}(r)$ for its entire journey, then

$$\Delta t = \int_{R(t)}^{0} \frac{\mathrm{d}r}{v_{\rm ff}(r)}$$
$$= \frac{-1}{\sqrt{GM_*}} \int_{R(t)}^{0} r^{1/2}\mathrm{d}r \qquad \text{Should be } \sqrt{2GM_*}$$
$$= \frac{2R^{3/2}}{3\sqrt{GM_*}}. \tag{6}$$

(b) We'll make some simplifying assumptions about the protostar and rarefaction wave. If we assume a constant accretion rate based on the cloud freefall speed,

$$M_* = \dot{M}t = \frac{a_T^3}{G}t.$$

We then parametrize R in terms of the sound speed and a factor f:

$$R \equiv f a_T t.$$

Plugging these definitions into (6), we find

$$\frac{\Delta t}{t} = \frac{2}{3} f^{3/2}.$$

This is negligible only for small values of f; that is, the steady-state assumption is only valid well inside the rarefaction wave.

(c) Equation (10.34) tells us that

$$\rho(r) = \frac{1}{4\pi\sqrt{2}} G^{-1/2} M_*^{-1/2} \dot{M} r^{-3/2}.$$

Calculating ΔM is an easy integral:

$$\Delta M = \int_0^R 4\pi r^2 \rho(r)\mathrm{d}r$$
$$= (2GM_*)^{-1/2}\dot{M} \int_0^R r^{1/2}\mathrm{d}r$$
$$= \frac{2}{3}(2GM_*)^{-1/2}\dot{M}R^{3/2}.$$

(d) As before, let $M_* = a_T^3 t/G$ and $R = f a_T t$. Then, substituting,

$$\frac{\Delta M}{M_*} = \frac{\sqrt{2}}{3} f^{3/2}.$$

As before, this is only negligibly small if $f \ll 1$, well inside the rarefaction wave.