AY250 Assignment 4

due: Thursday, Oct 21, 2010

1 - A star of mass M_* is embedded in a large, diffuse cloud. The sound speed in the isothermal gas is a_T, and its density far from the star is ρ_∞. Consider the steady-state, spherical accretion of gas onto the star. Neglecting the self-gravity of the gas, equations (10.28) and (10.29) yield

$$\begin{aligned} \dot{M} &= 4\,\pi\, r^2\, \rho\, u \\ u\,\frac{\partial u}{\partial r} &= -\frac{a_T^2}{\rho}\,\frac{\partial \rho}{\partial r} - \frac{G\,M_*}{r^2}\ . \end{aligned}$$

Here, we have let u denote the *infall* speed (the negative of our usual convention), and have set $\partial u/\partial t = 0$ in the momentum equation, as is appropriate in steady state. The problem is to derive the accretion rate $\dot{M}$.

(a) Define a nondimensional distance $x \equiv r\,a_T^2/(G\,M_*)$, a nondimensional velocity $y \equiv u/a_T$, and a nondimensional density $z \equiv \rho/\rho_\infty$. Finally, define a nondimensional accretion rate by

$$\lambda \equiv \frac{\dot{M}\,a_T^3}{4\,\pi\,G^2\,M_*^2\,\rho_\infty}\ .$$

Rewrite the mass continuity and momentum equations in terms of x, y, z, and λ.

(b) Eliminate z between these two equations to find an algebraic relation between y, x, and λ. To do this, you may assume that y vanishes at large x.

(c) There are many different solutions for $y(x)$, depending on the value of λ. In the physically relevant solution, y increases monotonically for decreasing x, diverging at $x = 0$. Find $\lambda_{\rm crit}$, the unique value of λ for which the solution behaves in this manner. Thus, derive a dimensional expression for $\dot{M}$.

2 - In the previous problem, we assumed a constant stellar mass M_*. However, as a result of the finite accretion rate $\dot{M}$, this mass will increase.

(a) Write a differential equation for $M_*(t)$. Your equation should contain, in addition to M_* and t, $\lambda_{\rm crit}$, G, ρ_∞, and a_T.

(b) Let $M_\circ$ be the initial stellar mass. Define a nondimensional mass $m \equiv M_*/M_\circ$, and a nondimensional time $\tau \equiv t\,\sqrt{G\,\rho_\infty}$. Finally, let β be the following nondimensional quantity:

$$\beta \equiv \frac{4\,\pi\,\lambda_{\rm crit}\,G^{3/2}\,M_\circ\,{\rho_\infty}^{1/2}}{a_T^3}\ .$$

Recast your differential equation as one for $m(\tau)$. The equation should only contain m, τ, and β.

(c) Solve the nondimensional equation analytically. What happens to $m(\tau)$ at late times?

(d) More generally, does this picture of steady-state flow apply to accreting protostars?

3- Consider how differently radiation propagates within the protostellar dust envelope and the opacity gap. In spherical geometry, for a radiation field that is azimuthally symmetric, the radiative transfer equation becomes

$$\mu \frac{\partial I_\nu}{\partial r} + \frac{\left(1-\mu^2\right)}{r} \frac{\partial I_\nu}{\partial \mu} = -\rho \kappa_\nu I_\nu + \rho \kappa_\nu B_\nu \ .$$

Here, μ is the cosine of the angle between the local direction of radiation and the outward, radial direction. On the righthand side of this equstion, we have assumed that the emission, here from dust, is thermal, and employed Kirkhoff's law. We shall be interested in the energy density, energy flux, and momentum flux (i.e., pressure) of the radiation field:

$$\begin{aligned} u_{\rm rad} &\equiv \frac{2\pi}{c} \int d\nu \int d\mu \, I_\nu \\ F_{\rm rad} &\equiv 2\pi \int d\nu \int d\mu \, \mu \, I_\nu \\ P_{\rm rad} &\equiv \frac{2\pi}{c} \int d\nu \int d\mu \, \mu^2 \, I_\nu \ . \end{aligned}$$

(a) Find an ordinary differential equation for the radial variation of $P_{\rm rad}$. Your equation will also contain $u_{\rm rad}$, $F_{\rm rad}$, and a flux-weighted mean opacity κ.

(b) Since $F_{\rm rad}$ is known from the luminosity, $L = L_{\rm acc}$, we may solve this equation if we know a relationship between $P_{\rm rad}$ and $u_{\rm rad}$. It is conventional to write

$$P_{\rm rad} = f \, u_{\rm rad} \ ,$$

where f is called the Eddington factor. Find f for the nearly isotropic field in the dust envelope. Assuming additionally that the radiation field is nearly blackbody, so that $u_{\rm rad} = a T^4$, convert your equation from (a) into one for $T(r)$, and compare to equation (11.9) in the text. (c) Turning to the opacity gap, the main sources of radiation are the gas photosphere at R_g, which bounds the radiative precursor, and hot grains at R_d, the dust destruction front. If we were to neglect emission from the latter and consider points far from R_g, what would f be?

(d) More realistically, we should include emission from the dust destruction front, although its temperature T_d is less than T_g at the gas photosphere. Taking the two surfaces to be blackbodies at their respective temperatures, derive a more general formula for f at any intermediate radius r. Verify that f has the expected behavior as (i) T_d approaches T_g, and (ii) $T_d \ll T_g$ and $r \gg R_g$. (*Hint:* Consider how I_ν varies over all possible μ-values at a fixed point within the opacity gap.)

4 - Protostars are fully convective from about 0.4 $M_\odot$ to 2.4 $M_\odot$. This range encompasses the majority of all stars, according to the initial mass function. It is therefore important to understand how stellar radii vary over this interval . According to Figure 11.21, protostars first swell from deuterium burning, and then shrink because of their self-gravity. A simple analytic model explains both effects.

(a) Integrate the heat equation (11.19) from the center of the star to the postshock relaxation point, where you may set $L_{\rm int} \approx 0$. Assume that deuterium is burning at the steady-state value given in equation (11.30). Recalling that the specific entropy s is a spatial constant, write down an expression for the mass integral of $T\, ds/dM_*$.

(b) Since we want to obtain a mass-radius relation, expand ds/dM_* in terms of partial derivatives at fixed R_* and M_*. From the theory of polytropes, to be introduced in Chapter 16, the partial derivatives are

$$\left(\frac{\partial s}{\partial M_*}\right)_{R_*} = \frac{\mathcal{R}}{2\,\mu\, M_*}$$
$$\left(\frac{\partial s}{\partial R_*}\right)_{M_*} = \frac{3\,\mathcal{R}}{2\,\mu\, R_*} \ .$$

By further using the relation derived in equations (16.25)-(16.27),

$$\int_0^{M_*} T\, dM_r = \frac{2\,\mu\, G\, M_*^2}{7\,\mathcal{R}\, R_*} \ ,$$

find a differential equation for R_* as a function of M_*.

(c) Simplify your equation by defining the nondimensional quantities

$$m \equiv \frac{M_*}{M_\circ}$$
$$r \equiv \frac{7\,\delta\, R_*}{4\, G\, M_\circ} \ ,$$

where $M_\circ$ is some fiducial mass. Integrate the nondimensional differential equation analytically, and show that $r(m)$ has the expected behavior.

5 - In 12.1, we briefly described density perturbations in a collapsing isothermal sphere. Let us generalize this result and place it in the larger context of adiabatic perturbations.

(a) Suppose that the pressure P varies as ρ^γ, where γ is the ratio of specific heats at constant pressure and volume. Using equation (9.23) together with the ideal gas law, how does the Jeans length λ_J vary with ρ?

(b) We may continue to assume that λ, the size of any perturbation of fixed mass, scales as $\rho^{-1/3}$. Find how the ratio λ/λ_J scales with density.

(c) Summarize the behavior you expect for $\gamma = 1, 4/3$, and $5/3$. Which perturbations grow and which die away?

6 - We have seen that insight can be gained into binary origins by considering the idealized problem of an infinite, self-gravitating, isothermal cylinder. Let us explore basic properties of this configuration. The equation of hydrostatic balance is expressed as a simple modification of equation (9.4):

$$\rho\,(\varpi) = \rho_c \exp\left(-\Phi_g/a_T^2\right) \; .$$

Poisson's equation (9.3) becomes, in cylindrical coordinates,

$$\frac{\partial^2 \Phi_g}{\partial \varpi^2} + \frac{1}{\varpi}\frac{\partial \Phi_g}{\partial \varpi} = 4\,\pi\,G\,\rho \; .$$

(a) Define the nondimensional radius $\xi \equiv (4\,\pi\,G\,\rho_c/a_T^2)^{1/2}\,\varpi$ and the nondimensional potential $\psi \equiv \Phi_g/a_T^2$. Show that the above equations combine to yield

$$\frac{d^2\psi}{d\xi^2} + \frac{1}{\xi}\frac{d\psi}{d\xi} = \exp\left(-\psi\right) \; ,$$

with boundary conditions $\psi(0) = \psi'(0) = 0$.

(b) Verify that the above equation is solved analytically by

$$\psi = 2\,\ln\left(1 + \xi^2/8\right) \; ,$$

and that this solution properly meets the two boundary conditions.

(c) Imagine that the cylinder is surrounded by a uniform pressure $P_\circ$. If $\varpi_\circ$ is the dimensional radius, define a new nondimensional one by $\tilde{\varpi}_\circ \equiv (G\,P_\circ/a_T^4)^{1/2}\,\varpi_\circ$. Find an algebraic expression for $\tilde{\varpi}_\circ$ as a function of the center-to-edge density contrast $x \equiv \rho_c\,a_T^2/P_\circ$.

(d) Show that $\tilde{\varpi}_\circ$ tends to zero when either x is small (*i.e.*, close to unity) or large. For what value of the density contrast is $\tilde{\varpi}_\circ$ a maximum? What is this maximum $\tilde{\varpi}_\circ$?