

AY 250 Assignment 4: Solutions

Problem 1

- (a) To rewrite the equations, we simply express the dimensional quantities in terms of the nondimensional values that we derived:

$$\begin{aligned}r &= GM_* a_T^{-2} x, \\u &= a_T y, \\ \rho &= \rho_\infty z, \\ \dot{M} &= 4\pi G^2 M_*^2 \rho_\infty a_T^{-3} \lambda.\end{aligned}$$

The only subtlety here is changing the derivatives:

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} = \frac{a_T^2}{GM_*} \frac{\partial}{\partial x}.$$

Unsurprisingly, once these substitutions are made, all of the constants drop out, and we're left with

$$\lambda = x^2 y z \quad (1)$$

and

$$y \frac{\partial y}{\partial x} = -z^{-1} \frac{\partial z}{\partial x} - x^{-2}. \quad (2)$$

- (b) This is a bit trickier than it looks. From (1), trivially $z = \lambda x^{-2} y^{-1}$, but plugging this into (2) causes the λ to drop out, which doesn't help us any. Instead, first integrate (2) to find

$$\frac{y^2}{2} = -\ln z + \frac{1}{x} + C.$$

As $x \rightarrow \infty$, z goes to unity by its definition and y goes to zero in the physically relevant solution. Thus, the integration constant C is zero. Subsequently, plugging into (1), we find

$$\lambda = x^2 y \exp(x^{-1} - y^2/2). \quad (3)$$

- (c) The second half of the trickiness comes in here. If we do indeed plug in $z = \lambda x^{-2} y^{-1}$ into (2), we obtain

$$y \frac{\partial y}{\partial x} = 2x^{-1} + y^{-1} \frac{\partial y}{\partial x} - x^{-2}.$$

With a little massaging, we can solve this for $\partial y/\partial x$ and get

$$\frac{\partial y}{\partial x} = \frac{2x-1}{y^2-1} \frac{y}{x^2}.$$

A physical solution should have finite (and positive) $\partial y/\partial x$ everywhere except at $x=0$. This means that at $y=1$, it must be true that $x=1/2$; otherwise the derivative would go to infinity there. And we know that $y(x)$ does pass through $y=1$ since, as the

problem suggests, $\lim_{x \rightarrow \infty} y = 0$ and $\lim_{x \rightarrow 0} y = \infty$. Since λ is a constant, we can just plug this pair of x and y values into (3) to get

$$\lambda_{\text{crit}} = \frac{e^{3/2}}{4}$$

and hence

$$\dot{M} = \pi e^{3/2} G^2 M_*^2 \rho_{\infty} a_T^{-3}.$$

Problem 2

- (a) If we assume all the infalling mass eventually lands on the protostar, we can identify dM_*/dt and \dot{M} , in which case the definition of λ implies

$$\frac{dM_*}{dt} = 4\pi G^2 M_*^2 \rho_\infty a_T^{-3} \lambda_{\text{crit}}.$$

- (b) We again reëxpress our dimensional quantities in terms of the dimensionless ones:

$$\begin{aligned} M_* &= M_0 m, \\ t &= (G\rho_\infty)^{-1/2} \tau. \end{aligned}$$

Again, we must take care with changing derivatives:

$$\frac{d}{dt} = \frac{d\tau}{dt} \frac{d}{d\tau} = (G\rho_\infty)^{1/2} \frac{d}{d\tau}.$$

- * As one might expect, when the substitutions are made, the parameter β pops out and we're left with:

$$\frac{dm}{d\tau} = \beta m^2. \quad (4)$$

- (c) We can easily separate the variables of (4) and integrate both sides to solve the differential equation:

$$\begin{aligned} \int \frac{dm}{m^2} &= \beta \int d\tau \\ -\frac{1}{m} &= \beta\tau + C \end{aligned}$$

Our boundary condition is that when $\tau = 0$, $m = 1$, so $C = -1$ and

$$m = \frac{1}{1 - \beta\tau} \quad (5)$$

If we define "late times" to be those when $\tau \rightarrow \beta^{-1}$, then we see that the denominator in (5) vanishes, and $m(\tau)$ diverges.

- (d) This picture does *not* apply to protostars, which accrete during the collapse of a self-gravitating cloud. Here, the self-gravity of the infalling material is completely ignored, which leads to the divergent behavior found in the previous part. Intuitively, the self-gravity of the cloud will brake the infalling material and prevent the runaway growth of the protostar suggested by this problem.

Problem 3

a)

Start with the given radiative transfer equation and multiply by μ :

$$\mu^2 \frac{\partial I_\nu}{\partial r} + \frac{(\mu - \mu^3)}{r} \frac{\partial I_\nu}{\partial \mu} = -\rho \kappa_\nu \mu I_\nu + \rho \kappa_\nu \mu B_\nu$$

Then, integrate over μ (make use of the isotropy of B_ν to set its integral over μ to zero):

$$\frac{\partial}{\partial r} \int_{-1}^1 \mu^2 I_\nu d\mu + \frac{1}{r} \int_{-1}^1 \mu \frac{\partial I_\nu}{\partial \mu} d\mu - \frac{1}{r} \int_{-1}^1 \mu^3 \frac{\partial I_\nu}{\partial \mu} d\mu = -\rho \kappa_\nu \int_{-1}^1 \mu I_\nu d\mu.$$

Use integration by parts to re-express the second and third terms.

$$\begin{aligned} \frac{1}{r} \int_{-1}^1 \mu \frac{\partial I_\nu}{\partial \mu} d\mu &= \frac{1}{r} \left[\int_{-1}^1 \frac{\partial}{\partial \mu} (\mu I_\nu) d\mu - \int_{-1}^1 I_\nu d\mu \right] \\ &= \frac{1}{r} \left[I_\nu(\mu=1) + I_\nu(\mu=-1) - \int_{-1}^1 I_\nu d\mu \right] \\ \frac{1}{r} \int_{-1}^1 \mu^3 \frac{\partial I_\nu}{\partial \mu} d\mu &= \frac{1}{r} \left[\int_{-1}^1 \frac{\partial}{\partial \mu} (\mu^3 I_\nu) d\mu - \int_{-1}^1 3\mu^2 I_\nu d\mu \right] \\ &= \frac{1}{r} \left[I_\nu(\mu=1) + I_\nu(\mu=-1) - 3 \int_{-1}^1 \mu^2 I_\nu d\mu \right] \end{aligned}$$

Note the terms that cancel and the radiative transfer equation becomes

$$\frac{\partial}{\partial r} \int_{-1}^1 \mu^2 I_\nu d\mu - \frac{1}{r} \int_{-1}^1 I_\nu d\mu + \frac{1}{r} 3 \int_{-1}^1 \mu^2 I_\nu d\mu = -\rho \kappa_\nu \int_{-1}^1 \mu I_\nu d\mu.$$

The next step is to multiply by 2π , integrate over frequency, and replace the terms with u_{rad} , F_{rad} , and P_{rad} . In the last step we define the mean opacity $\kappa = \frac{\int \kappa_\nu F_\nu d\nu}{F_{\text{rad}}}$, where $F_\nu \equiv 2\pi \int_{-1}^1 \mu I_\nu d\mu$. Then the final differential equation for the radial variation of P_{rad} is

$$\frac{\partial P_{\text{rad}}}{\partial r} + 3 \frac{P_{\text{rad}}}{r} - \frac{u_{\text{rad}}}{r} = -\frac{\rho \kappa}{c} F_{\text{rad}}.$$

b)

For isotropic I_ν we have

$$P_{\text{rad}} = \frac{2\pi}{c} \int I_\nu d\nu \int_{-1}^1 \mu^2 d\mu = \frac{2\pi}{c} \int I_\nu d\nu \frac{2}{3} = \frac{1}{3} \frac{4\pi}{c} \int I_\nu d\nu$$

$$u_{\text{rad}} = \frac{2\pi}{c} \int I_\nu d\nu \int_{-1}^1 d\mu = \frac{4\pi}{c} \int I_\nu d\nu$$

Therefore $P_{\text{rad}} = \frac{1}{3} u_{\text{rad}}$ and $f = \frac{1}{3}$.

We apply $P_{\text{rad}} = \frac{1}{3} u_{\text{rad}}$ to the resultant differential equation from part (a) to acquire

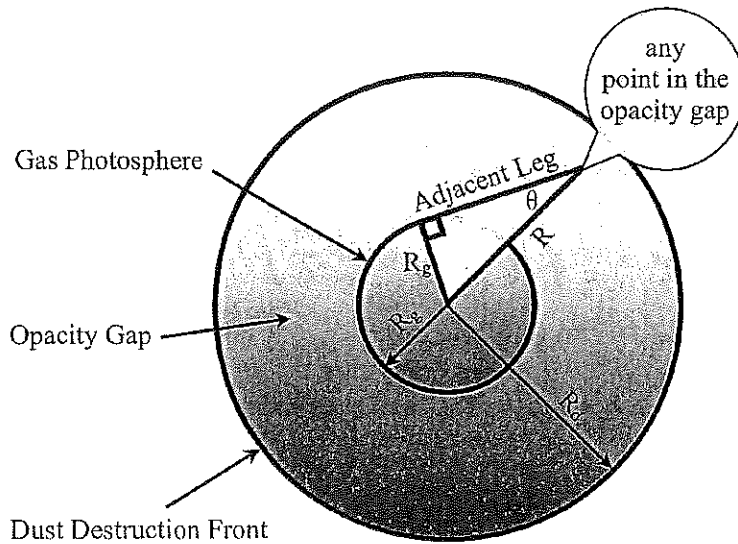
$$\frac{1}{3} \frac{\partial u_{\text{rad}}}{\partial r} = -\frac{\rho\kappa}{c} F_{\text{rad}}.$$

We make the substitutions $F_{\text{rad}} = \frac{L_{\text{acc}}}{4\pi r^2}$ and $\frac{\partial u_{\text{rad}}}{\partial r} = \frac{16\sigma T^3}{c} \frac{\partial T}{\partial r}$ to arrive at an equation for $T(r)$ which is identical to text equation (11.9):

$$T^3 \frac{\partial T}{\partial r} = -\frac{3\rho\kappa L_{\text{acc}}}{64\pi\sigma r^2}$$

c)

At any point in the opacity gap we have a right triangle formed with the opposite leg R_g , the hypotenuse R (distance from the center), and then (by the Pythagorean Theorem) the adjacent leg $\sqrt{R^2 - R_g^2}$. Thus, $\mu \equiv \cos\theta$ is constrained to lie between μ_o and 1, where $\mu_o = \frac{\sqrt{R^2 - R_g^2}}{R} = \sqrt{1 - \left(\frac{R_g}{R}\right)^2}$. See the following figure for an illustration of the geometry.



We can then write out P_{rad} and u_{rad} :

$$P_{\text{rad}} = \frac{2\pi}{c} \int B_\nu d\nu \int_{\mu_0}^1 \mu^2 d\mu$$

$$= \frac{2\pi}{3c} (1 - \mu_0^3) \int B_\nu d\nu$$

$$u_{\text{rad}} = \frac{2\pi}{c} \int B_\nu d\nu \int_{\mu_0}^1 d\mu$$

$$= \frac{2\pi}{c} (1 - \mu_0) \int B_\nu d\nu$$

$$\Rightarrow f = \frac{P_{\text{rad}}}{u_{\text{rad}}} = \frac{1}{3} \frac{(1 - \mu_0^3)}{(1 - \mu_0)} = \frac{1}{3} \frac{\left(1 - \left[1 - \left(\frac{R_g}{R}\right)^2\right]^{\frac{3}{2}}\right)}{\left(1 - \left[1 - \left(\frac{R_g}{R}\right)^2\right]^{\frac{1}{2}}\right)}$$

Since $\left(\frac{R_g}{R}\right)^2$ is small, we can Taylor expand $\left[1 - \left(\frac{R_g}{R}\right)^2\right]^n \approx 1 - n \left(\frac{R_g}{R}\right)^2$.

The result:

$$f = \frac{1}{3} \frac{1 - 1 - \frac{3}{2} \frac{R_g^2}{R^2}}{1 - 1 - \frac{1}{2} \frac{R_g^2}{R^2}} = \frac{1}{3} \frac{\frac{3}{2}}{\frac{1}{2}} = 1.$$

d)

The general approach is similar to that in (c), except this time we need to keep track of $B_\nu(T_g)$ and $B_\nu(T_d)$. Also, the geometrical limits for the integrals with respect to μ for emission from the dust destruction front are -1 to μ_o .

$$\begin{aligned}
 P_{\text{rad}} &= \frac{2\pi}{c} \int B_\nu(T_g) d\nu \int_{\mu_o}^1 \mu^2 d\mu + \frac{2\pi}{c} \int B_\nu(T_d) d\nu \int_{-1}^{\mu_o} \mu^2 d\mu \\
 &= \frac{2\pi}{3c} (1 - \mu_o^3) \int B_\nu(T_g) d\nu + \frac{2\pi}{3c} (1 + \mu_o^3) \int B_\nu(T_d) d\nu \\
 &= \frac{2\pi}{3c} \left[\frac{acT_g^4}{4\pi} (1 - \mu_o^3) + \frac{acT_d^4}{4\pi} (1 + \mu_o^3) \right] \\
 &= \frac{a}{6} [T_g^4 (1 - \mu_o^3) + T_d^4 (1 + \mu_o^3)]
 \end{aligned}$$

Likewise we find,

$$u_{\text{rad}} = \frac{a}{2} [T_g^4 (1 - \mu_o) + T_d^4 (1 + \mu_o)].$$

Thus, we have for the Eddington factor

$$f = \frac{1}{3} \times \frac{[T_g^4 (1 - \mu_o^3) + T_d^4 (1 + \mu_o^3)]}{[T_g^4 (1 - \mu_o) + T_d^4 (1 + \mu_o)]}.$$

Simplify by expanding terms and defining $D \equiv \frac{(T_g^4 - T_d^4)}{(T_g^4 + T_d^4)}$. The final relation is

$$f = \frac{1}{3} \times \frac{1 - \mu_o^3 D}{1 - \mu_o D}.$$

As $T_d \rightarrow T_g$, $D \rightarrow 0$ and then $f \rightarrow \frac{1}{3}$. This is the isotropic blackbody field since the surfaces emitting radiation are all of the same temperature.

For $T_d \ll T_g$ and $r \gg R_g$, $D \rightarrow 1$ and we recover the formula from part (c) with $f \rightarrow 1$.

Problem 4

a)

Text equation (11.19):

$$\frac{\partial L_{\text{int}}}{\partial M_r} = \epsilon_D - T \frac{\partial s}{\partial t}.$$

Integrating,

$$\int dL_{\text{int}} = \int_0^{M_*} \epsilon_D dM_r - \int_0^{M_*} T \frac{\partial s}{\partial t} dM_r.$$

Note that the left side is zero (luminosity at the center is zero because the enclosed mass is zero and luminosity at the postshock relaxation point is given as zero) and that the first term on the right is $L_D = \dot{M}\delta$ by equation (11.30). Then we have

$$\dot{M}\delta = \int_0^{M_*} T \frac{\partial s}{\partial t} dM_r.$$

Since s is a spatial constant at any time, $\frac{\partial s}{\partial t} = \dot{M} \frac{ds}{dM_*}$. We may then rewrite the expression for δ ,

$$\delta = \int_0^{M_*} T \frac{ds}{dM_*} dM_r.$$

b)

$$\frac{ds}{dM_*} = \left(\frac{\partial s}{\partial R_*} \right)_{M_*} \frac{dR_*}{dM_*} + \left(\frac{\partial s}{\partial M_*} \right)_{R_*}$$

Applying the supplied relations for the partial derivatives (from the theory of polytropes, Chapter 16) yields

$$\frac{ds}{dM_*} = \frac{3\mathcal{R}}{2\mu R_*} \frac{dR_*}{dM_*} + \frac{\mathcal{R}}{2\mu M_*}.$$

Plug into our result from (a):

$$\begin{aligned} \delta &= \int_0^{M_*} \frac{\mathcal{R}}{2\mu M_*} T dM_r + \int_0^{M_*} \frac{3\mathcal{R}}{2\mu R_*} \frac{dR_*}{dM_*} T dM_r \\ \delta &= \frac{3\mathcal{R}}{2\mu R_*} \frac{dR_*}{dM_*} \int_0^{M_*} T dM_r + \frac{\mathcal{R}}{2\mu M_*} \int_0^{M_*} T dM_r. \end{aligned}$$

Then, with the supplied polytropic identity (from text equations (16.25-27)),

$$\delta = \frac{3\mathcal{R}}{2\mu R_*} \frac{dR_*}{dM_*} \frac{2\mu GM_*^2}{7\mathcal{R}R_*} + \frac{\mathcal{R}}{2\mu M_*} \frac{2\mu GM_*^2}{7\mathcal{R}R_*}$$

$$\delta = \frac{3GM_*^2}{7R_*^2} \frac{dR_*}{dM_*} + \frac{GM_*}{7R_*}.$$

Then the differential equation can be expressed as

$$\frac{dR_*}{dM_*} = \frac{7\delta R_*^2}{3GM_*^2} - \frac{R_*}{3M_*}.$$

c)

We define

$$m \equiv M_*/M_\odot$$

$$r \equiv \frac{7\delta R_*}{4GM_\odot}.$$

Plugging into the final differential equation from part (b) and simplifying, we have

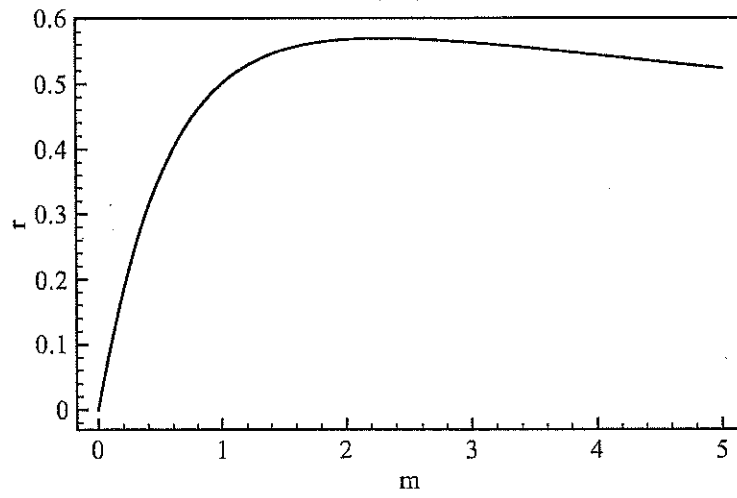
$$\frac{dr}{dm} = \frac{1}{3} \left(4 \frac{r^2}{m^2} - \frac{r}{m} \right).$$

Solving we find

$$r(m) = \frac{m}{1 + C_1 m^{\frac{4}{3}}}.$$

where C_1 is a constant to be determined by boundary conditions. This shows the expected behavior of increasing r for low m which eventually turns over as m grows large enough for the protostar to contract with increased mass. Below we plot $r(m)$ for $C_1 = 1$.

Plot for 5c



Problem 5

a)

From (9.23),

$$\lambda_J = \left(\frac{\pi a_T^2}{G\rho} \right)^{1/2}$$

and the ideal gas law,

$$\begin{aligned} P &\sim \rho a_T^2 \sim \rho^\gamma \\ \rightarrow a_T^2 &\sim \rho^{\gamma-1} \end{aligned}$$

We find

$$\begin{aligned} \lambda_J &\sim a_T \rho^{-1/2} \\ &\sim \rho^{(\gamma-2)/2} \end{aligned}$$

b)

For $\lambda \sim \rho^{-1/3}$,

$$\begin{aligned} \frac{\lambda}{\lambda_J} &\sim \frac{\rho^{-1/3}}{\rho^{(\gamma-2)/2}} \\ \Rightarrow \frac{\lambda}{\lambda_J} &\sim \rho^{(4-3\gamma)/6} \end{aligned}$$

c)

For $\gamma = 1$, $\frac{\lambda}{\lambda_J} \sim \rho^{1/6} \rightarrow$ the perturbation grows

For $\gamma = 4/3$, $\frac{\lambda}{\lambda_J} \sim \rho^0 \rightarrow$ the wave is stable

For $\gamma = 5/3$, $\frac{\lambda}{\lambda_J} \sim \rho^{-1/6} \rightarrow$ the perturbation dies away

Problem 6

- (a) Once again we substitute nondimensional parameters for our dimensional variables. Trivially,

$$\rho = \rho_c \exp(-\psi). \quad (10)$$

To recast the Poisson equation, we once again must convert derivatives:

$$\frac{\partial}{\partial \varpi} = \frac{\partial \xi}{\partial \varpi} \frac{\partial}{\partial \xi} = (4\pi G \rho_c a_T^{-2})^{1/2} \frac{\partial}{\partial \xi},$$

so the Poisson equation becomes

$$4\pi G \rho_c a_T^{-2} \left(\frac{\partial^2 (a_T^2 \psi)}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial (a_T^2 \psi)}{\partial \xi} \right) = 4\pi G \rho_c \exp(-\psi),$$

which reduces to the desired expression

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial \psi}{\partial \xi} = \exp(-\psi). \quad (11)$$

- (b) First off, clearly $\psi(0) = 0$. Then

$$\psi'(\xi) = \frac{2}{1 + \xi^2/8} \cdot \frac{\xi}{4} = \frac{\xi}{2 + \xi^2/4},$$

in which case $\psi'(0) = 0$ as well. Taking the second derivative,

$$\begin{aligned} \psi''(\xi) &= \frac{(2 + \xi^2/4) - \xi^2/2}{(2 + \xi^2/4)^2} \\ &= \frac{2 - \xi^2/4}{(2 + \xi^2/4)^2}. \end{aligned}$$

Some straightforward algebra reveals that both sides of (11) evaluate to $(1 + \xi^2/8)^{-2}$, so the given function is indeed a solution.

- (c) The boundary of the cylinder is implicitly set to be the location where its pressure is equal to P_0 ; that is,

$$\rho(\varpi_0) a_T^2 = P_0,$$

in which case

$$\rho(\varpi_0) = \frac{\rho_c}{x}.$$

From (10) and our chosen expression for ψ , we find

$$\frac{\rho_c}{x} = \frac{\rho_c}{(1 + \xi_0^2/8)^2}$$

where ξ_0 is the value of ξ corresponding to ϖ_0 . Putting everything in terms in $\tilde{\varpi}_0$, we find

$$\xi_0 = (4\pi x)^{1/2} \tilde{\varpi}_0,$$

so our expression can be reduced to

$$x = \left(1 + \frac{\pi}{2}x\tilde{\omega}_0^2\right)^2.$$

Solving for $\tilde{\omega}_0^2$,

$$\tilde{\omega}_0^2 = \frac{2\sqrt{x}-1}{\pi x}. \quad (12)$$

- (d) For large x , (12) clearly goes to zero. By its definition, the smallest reasonable value of x is unity, in which case $\tilde{\omega}_0^2 = 0$ as well.

To find where $\tilde{\omega}_0$ reaches its maximum and what value it attains, we just need to do some old-school calculus. Since $d\tilde{\omega}_0^2/dx = 2\tilde{\omega}_0 d\tilde{\omega}_0/dx$, a maximum of $\tilde{\omega}_0^2$ is also a maximum of $\tilde{\omega}_0$, which makes life a bit easier. The maximum occurs at

$$\begin{aligned} 0 &= \frac{d\tilde{\omega}_0^2}{dx} \\ &= \frac{2}{\pi} \frac{d}{dx} \left(x^{-1/2} - x^{-1} \right) \\ &= \frac{2}{\pi} \left(-\frac{x^{-3/2}}{2} + x^{-2} \right), \end{aligned}$$

which is solved by $x = 4$. Plugging in, the maximal value is

$$\tilde{\omega}_{0,\max} = (2\pi)^{-1/2}.$$