Solutions to Star Formation Homework 5, Assigned by Steven Stahler

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Problem 1

Part a

By definition

$$\frac{M_f}{M_i} = \epsilon \tag{1}$$

Using equation 12.37

$$\boxed{\frac{R_f}{R_i} = \frac{M_i}{M_f} = 1/\epsilon} \tag{2}$$

The virial theorem tells us

$$\frac{1}{2}Mv^2 = \frac{\eta G M^2}{2R} \tag{3}$$

This directly leads to

$$v \propto \sqrt{\frac{M}{R}} \tag{4}$$

and so

$$\boxed{\frac{v_f}{v_i} = \sqrt{\frac{M_f}{M_i}\frac{R_i}{R_f}} = \epsilon} \tag{5}$$

By definition

$$\rho \propto \frac{M}{R^3} \tag{6}$$

So

$$\boxed{\frac{\rho_f}{\rho_i} = \frac{M_f}{M_i}\frac{R_i^3}{R_f^3} = \epsilon^4} \tag{7}$$

So when epsilon = 0.3, we get

$$\frac{R_f}{R_i} = 3.33 \qquad \frac{v_f}{v_i} = 0.3 \qquad \frac{\rho_f}{\rho_i} = 0.0081 \tag{8}$$

Part b

The mass ejected ΔM can be expressed by

$$\Delta M = M_f - M_i = \epsilon M_i - M_i = M_i(\epsilon - 1) \tag{9}$$

Then from equation 12.40 we have

$$\boxed{\frac{R_f}{R_i} = \frac{M_i + M_i(\epsilon - 1)}{M_i + 2[M_i(\epsilon - 1)]} = \frac{\epsilon}{2\epsilon - 1}} \tag{10}$$

We can once again use the proportionality from expression 4, and the last result for the radii, to get

$$\boxed{\frac{v_f}{v_i} = \sqrt{\frac{M_f}{M_i}\frac{R_i}{R_f}} = \sqrt{2\epsilon - 1}} \tag{11}$$

For the densities we use expression 6 once more to get

$$\boxed{\frac{\rho_f}{\rho_i} = \frac{M_f}{M_i}\frac{R_i^3}{R_f^3} = \frac{(2\epsilon - 1)^3}{\epsilon^2}} \tag{12}$$

So when epsilon = 0.6, we get

$$\frac{R_f}{R_i} = 3. \qquad \frac{v_f}{v_i} = 0.447 \qquad \frac{\rho_f}{\rho_i} = 0.022 \tag{13}$$

Part c

From the data given we have $v_f/v_i = 1/3$. If the cluster formed slowly then for our model it would obey $v_f/v_i = \epsilon$ and so we get $\epsilon = 1/3$. If the cluster formed rapidly then for our model it would obey $v_f/v_i = \sqrt{2\epsilon - 1}$ and so we get $\epsilon = 5/9$. Observations tell us that the value of epsilon corresponding to star forming regions such as Taurus is only about a few percent, indicating that our simplified model must be refined.

Problem 2

Part a

By the chain rule

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = -\frac{v}{t_0} \tag{14}$$

So we have a separable differential equation that can be integrated as follows:

$$\int_0^{t'} \frac{dv}{v} = \int_0^{t'} -\frac{dt}{t_0} \implies \ln v = -\frac{t}{t_0} + C \tag{15}$$

Exponentiating both sides of the last equality and using the condition $v(0) = v_0$ gives

$$v = v_0 e^{-\frac{t}{t_0}} \tag{16}$$

Part b

We can integrate the velocity from time 0 to infinity to get the asymptotic distance traveled:

$$r_\infty = \int_0^\infty v dt = v_0 \int_0^\infty e^{-t/t_0} dt \tag{17}$$

Making the change of variables $x = t/t_0$ we obtain

$$r_\infty = v_0 t_0 \int_0^\infty e^{-x} dt = v_0 t_0 \tag{18}$$

Part c

Once again we obtain an expression for position by integrating velocity

$$r = v_0 \int_0^{t'} e^{-t/t_0} dt = v_0 t_0 \big(- e^{-t'/t_0} + 1 \big) \tag{19}$$

Now we use the definition of t' in terms of θ and ω, along with our result from part a, to obtain

$$\frac{r}{r_\infty} = 1 - \exp\left[- \frac{1}{t_0} \Big(t - \frac{\theta}{\omega} \Big) \right] = 1 - \exp\left[\frac{1}{\omega t_0} \Big(\theta - \omega t \Big) \right] \tag{20}$$

In terms of the dimensionless variables this can be written as

$$r' = 1 - e^{\gamma \theta'} \tag{21}$$

Part d

Figure 1 plots $r'(\theta')$ for $\gamma = 5$ and for θ ranging between $-7\pi/24$ and 0, including the reflected branch. We see that this provides a decent match to the image of HH 34 in the textbook.

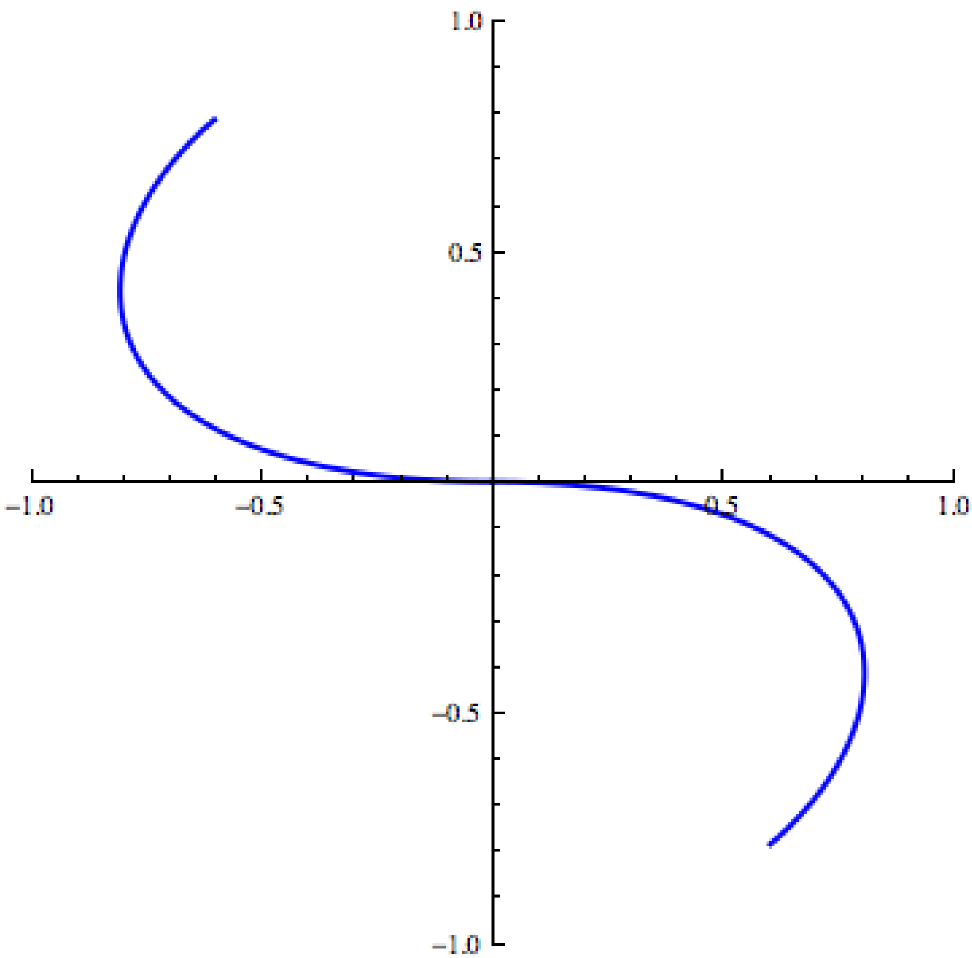

Figure 1: Our model for HH34, with $\gamma = 5$. The tick marks are in units of r_∞

Part e

Using the definitions of γ and v_0 we can solve for ω to obtain

$$\omega = \frac{v_0}{\gamma r_\infty} \tag{22}$$

But then since $\omega = 2\pi/T_0$, we have

$$T_0 = \frac{2\pi\gamma r_\infty}{v_0} \approx 10^5 \text{ years} \tag{23}$$

where we used $\gamma = 5$ from our result in part d.

Problem 3

Part a

We know that $R \propto M_*$ in this case, so I will call the constant of proportionality α, i.e., $R_* = \alpha M_*$. It is also true that $V_{\text{eq}} = R_*\Omega$. Making these substitutions into the given expression for Ω, and rearranging, yields

$$V_{\text{eq}} = f\sqrt{\frac{G}{\alpha}} \tag{24}$$

So V_{eq} is approximately constant and independent of mass, as claimed.

Part b

Making the substitutions for R_* and using the expression for V_{eq} from part a, we obtain

$$J_* = \alpha\beta M_*^2 V_{\text{eq}} \tag{25}$$

Part c

Let us assume that as mass flows in it accumulates on the outer radius of the star, so that for a little bit of added mass dM, its moment of inertia the instant it is added is dMR^2. Let us also assume that this mass dM is moving at speed V_{eq} around the center of the star. So then the angular momentum added is dMR_*V_{eq} and the rate of angular momentum being accumulated is $\dot{M}_{\text{in}}R_*V_{\text{eq}}$ or $\dot{M}_{\text{in}}V_{\text{eq}}\alpha M_*$

Part d

We begin with

$$\dot{J}_* = \dot{J}_{\text{in}} - \dot{J}_{\text{out}} \tag{26}$$

We can differentiate both sides of our result from part b with respect to time to obtain

$$\dot{J}_* = 2M_*\dot{M}_*V_{\text{eq}}\alpha\beta \tag{27}$$

We also have the relations

$$\dot{M}_* = \dot{M}_{\text{in}} - \dot{M}_{\text{out}} \tag{28}$$

$$\dot{J}_{\text{in}} = \dot{M}_{\text{in}}V_{\text{eq}}\alpha M_* \tag{29}$$

$$\dot{J}_{\text{out}} = \gamma\Omega_* R_*^2\dot{M}_{\text{out}} = \alpha\gamma V_{\text{eq}}M_*\dot{M}_{\text{out}} \tag{30}$$

Putting everything together we obtain

$$2M_*(\dot{M}_{\rm in} - \dot{M}_{\rm out})V_{\rm eq}\alpha\beta = \dot{M}_{\rm in}V_{\rm eq}\alpha M_* - \alpha\gamma V_{\rm eq}M_*\dot{M}_{\rm out} \tag{31}$$

After canceling, simplifying, and rearranging, we finally obtain

$$\dot{M}_{\rm in} = \frac{2\beta - \gamma}{2\beta - 1}\dot{M}_{\rm out} \tag{32}$$

Note that both the numerator and the denominator in the fraction multipilying $\dot{M}_{\rm out}$ are negative. With that in mind, in order to have $\dot{M}_{\rm in} > \dot{M}_{\rm out}$, we must have $\gamma > 1$.

Problem 4

Part a

Our simplified expression for $\bar{J}_s$ is

$$\bar{J}_s = \frac{\Gamma}{2B_{\rm ul}} \tag{33}$$

We can then use the relation between intensity and mean flux to write

$$I_s = \frac{4\pi\bar{J}_s}{\Delta\Omega} = \frac{2\pi\Gamma}{B_{\rm ul}\Delta\Omega} \tag{34}$$

Part b

Starting with equation 14.1 and plugging in our expression for I from part a we obtain

$$T_s = \frac{c^2\pi\Gamma}{\nu_0^2 k_{\rm B}B_{\rm ul}\Delta\Omega} \tag{35}$$

We now use the Einstein relation relating $A_{\rm ul}$ to $B_{\rm ul}$, as found in e.g. appendix B of the text:

$$A_{\rm ul} = \frac{2h\upsilon_0^3}{c^2}B_{\rm ul} \tag{36}$$

After making the substitution and simplifying, we obtain

$$T_s = \left(\frac{4\pi}{\Delta\Omega}\right)\left(\frac{\Gamma}{2A_{\rm ul}}\right)\left(\frac{h\nu_0}{k_{\rm B}}\right) \tag{37}$$

Part c

After rewriting $\Delta\Omega$ in terms of s and d, and solving for s, we obtain

$$s = \sqrt{\left(\frac{k_{\rm B}T_s}{h\upsilon_0}\right)\left(\frac{2A_{\rm ul}}{\Gamma}\right)\left(\frac{d^2}{4\pi}\right)} \tag{38}$$

Now we use $\nu_0 = 1665$ Mhz, so that $h\nu_0/k_{\rm B}$ is 0.0799 Kelvin. We also have $\Gamma/2A_{\rm ul} = 2.11 \times 10^8$, $T_s = 10^{13}$ Kelvin and $d = 10^{14}$ cm. As a result, we compute

$$s \approx 2.2 \times 10^{16} \text{ cm} \tag{39}$$

This is a plausible distance, since a typical HII region (where OH masers are produced) is at least 100 times larger.

Part d

This time we use $\nu_0 = 22$ Ghz, so $h\nu_0/k_{\rm B}$ is 1.06 Kelvin. We have $\Gamma/2A_{\rm ul} = 2.63 \times 10^8$, $T_s = 10^{14}$ Kelvin and $d = 10^{13}$ cm. So we compute s to be

$$s \approx 1.7 \times 10^{15} \text{ cm} \tag{40}$$

Again, this is a plausible aamplification distance, since it is about the diameter of a stellar jet, the environment for many H_2O masers.

Problem 5

Part a

For two brightness temperatures T_1 and T_2, we can use equation 14.1 to derive the following relation:

$$\frac{T_2}{T_1} = \frac{I_2}{I_1} \tag{41}$$

In this case we can set $T_1 = 3$ K and $T_2 = 3 \times 10^{15}$ K. Then $T_2/T_1 = 10^{15}$. The maser gain (as per the definition on page 503) is then $\ln T_2/T_1 = \ln 10^{15} = 34.5$.

Part b

From equation 14.26 we see that $\alpha_0 s$ is equal to the maser gain. So then using equation 14.28 we can write

$$\frac{\Delta\nu(s)}{\Delta\nu(0)} = \left(\ln \left[\frac{I(s)}{I(0)} \right] \right)^{-1/2} \tag{42}$$

So the line has narrowed by a factor of approximately 5.9, and the linewidth at the source is approximately 2.9 km s^{-1} .

Part c

For particles in thermal equilibrium at temperature T, the RMS speed in any given direction is $\sqrt{k_{\rm B}T/\mu}$ where μ is the mean mass per particle. For water μ is approximately 3×10^{-23} grams. Setting the speed from part b equal to this thermal speed gives a temperature of about 1.8×10^4 Kelvin. This temperature is not realistic since water only forms in the cooling regions of shocks at about 500 Kelvin. According to the text (p. 492), the expected linewidth at the source is close to the observed one of 0.5 kilometers per second. What happens is that the line begins to narrow, and then broadens, back to its original value as a result of saturation (see p. 504).

Problem 6

Part a

Geometrical reasoning leads to the relation

$$\cos[\pi/2 - (\theta + \Delta\theta))] = \frac{R\Delta\theta}{\Delta L} \tag{43}$$

We then solve for ΔL to first order in $\Delta\theta$ to obtain

$$\Delta L = \frac{R\Delta\theta}{\sin\theta} \tag{44}$$

Part b

The line-of-sight velocity $V_{\|}$ is equal to $V_\infty \cos\theta$. Thus, the difference in line-of-sight velocities measured at angular positions θ and $\theta + \Delta\theta$ is

$$V_\infty[\cos(\theta + \Delta\theta) - \theta] = V_\infty \Delta\theta \sin\theta \tag{45}$$

where once again we have kept the answer to first order in $\Delta\theta$. Setting this equal to the Doppler width yields

$$\Delta\theta = \frac{\Delta v_D}{V_\infty}\frac{1}{\sin\theta} \tag{46}$$

Part c

Substituting our expression for $\Delta\theta$ from part b into our expression from part a yields

$$\Delta L = R\frac{\Delta v_D}{V_\infty}\frac{1}{\sin^2\theta} = R\frac{\Delta v_D}{V_\infty}\frac{1}{1-\cos^2\theta} \tag{47}$$

But we also know

$$\cos\theta = \frac{V_r}{V_\infty} \tag{48}$$

So we can rearrange to obtain

$$\frac{\Delta L}{R} = \frac{\Delta\nu_D}{V_\infty}\frac{1}{1-(V_r/V_\infty)^2} \tag{49}$$

Figure 2 shows a plot of $\Delta L/R$ vs. V_r/V_∞. The observed spectrum should be proportional to what is shown in figure 2, because when the maser emission is saturated its intensity goes up linearly with the path length over which it is amplified, and because the observed emission frequencies will be Doppler shifted by an amount proportional to how V_r changes.

Part d

We can use our result from part b to write $\Delta v_D/V_\infty = \sin\theta\Delta\theta$. Substituting this expression into our answer from part c gives

$$\frac{\Delta L}{\Delta\theta} = \frac{R\sin\theta}{1-(V_r/V_\infty)^2} \tag{50}$$

But we also know from part a that

$$\frac{\Delta L}{\Delta\theta} = \frac{R}{\sin\theta} \tag{51}$$

So combining the last two equations yields

$$\sin^2\theta = 1 - (V_r/V_\infty)^2 \tag{52}$$

For a given θ, the observer sees a ring of radius $a(\theta) = R\sin\theta$, and so

$$a(V_r) = R\sqrt{1-(V_r/V_\infty)^2} \tag{53}$$

Part d

If we let d represent the distance to the maser and $\Delta\phi$ represent its angular diameter as seen by the observer, then we have the relation

$$(1/2)d\Delta\phi = R\sqrt{1-(V_r/V_\infty)^2} \tag{54}$$

We are given values for d, $\Delta\phi$, and V_r. We also know that V_∞ will be half of the value of the velocity difference of the two maser peaks. Solving for R and plugging in all the values gives a radius of approximately 8×10^{16} cm.

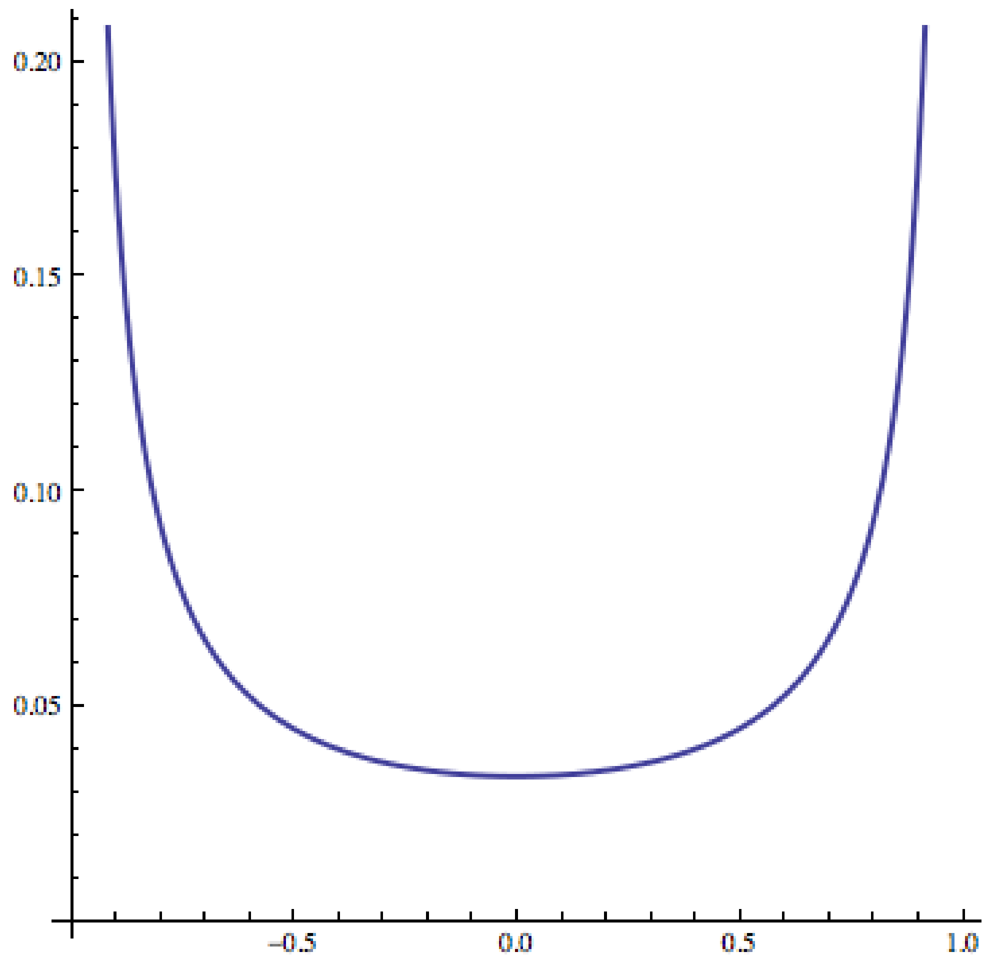

Figure 2: The vertical axis is $\Delta L/R$, the horizontal axis is V_r/V_∞