AY250 Assignment 6

due: Thursday, Nov 18, 2010

1 - The first expansion phase of an HII region ends once the high pressure of ionized gas drives a shock wave into the surrounding cloud. This happens when the speed of the expanding ionization front falls close to the sonic value. Let us derive the condition more quantitatively.

(a) In the frame of the ionization front, let $\rho_\circ$, $a_\circ$, and $u_\circ$ be the mass density, isothermal sound speed, and velocity, respectively, of neutral gas approaching the front. Similarly, let ρ_1, a_1, and u_1 be the analogous quantities in the ionized gas. By invoking mass and momentum conservation across the ionization front, derive a quadratic equation for $u_\circ/u_1$.

(b) Solving this quadratic equation, show that *either* $u_\circ \leq u_-$ *or* $u_\circ \geq u_+$. Here, u_- and u_+ are two characteristic speeds that can be expressed in terms of $a_\circ$ and a_1. Show further that $u_+ \, u_- = a_\circ^2$.

(c) The shock first appears when $u_\circ$ falls to u_+. At this point, the ionization and shock fronts are moving at the same speed. For $a_1 \gg a_\circ$, find an approximate expression for u_+. The statement that both fronts move at speed u_+ into the static, external medium is a refined version of the condition stated at the beginning of the problem.

(d) Because gas must now cross the shock, it enters the ionization front at a speed *lower* than u_+. We need to check that this speed is allowed by the inequalities you derived in (b). Let u_* be the postshock speed in our reference frame, and ρ_* the postshock density. Using the jump conditions for an isothermal shock (see Appendix B), find u_* in terms of $a_\circ$ and a_1. Thereby show how the inequalities you proved in part (b) are still satisfied.

2 - The terminal velocity of a radiatively accelerated wind is an appreciable fraction of the star's escape speed. To see why, note first, from Section 15.3, that the force due to a single line is proportional to the flux $F_\circ(r)$ times the combination $(r^2\, u)\, du/dr$. The force due to many lines may be approximated by a power law:

$$f_{\rm rad} \,=\, \frac{C}{r^2}\left(r^2\, u\, \frac{du}{dr}\right)^{\alpha} \quad .$$

The dimensional constant C is related to the mass loss rate. The nondimensional α, which lies between 0 and 1, depends on the optical thickness of the lines.

(a) Insert $f_{\rm rad}$ into the momentum equation (13.11), and neglect the thermal pressure gradient. As was the case with a single line, the combination $(r^2\, u)\, du/dr$ must be a constant, which we shall denote D. Find an algebraic equation relating C, D, and M_*.

(b) For a given value of α and C, there may be zero, one, or two D-values that satisfy your equation. Find a relation between C and D that guarantees a single, unique solution.

(c) Using this relation, find a differential equation for $u(r)$. Solve the equation, assuming $u(R_*) = 0$, where R_* is the star's radius.

(d) Find the relation between V_∞ and the stellar escape speed for the typical value $\alpha = 0.5$.

3 - In deriving the mass loss rate from a photoevaporating globule, we assumed that all of the incident, ionizing flux is absorbed at the base of the evaporative wind. We now investigate the situation more carefully.

Suppose a globule of radius R is located a distance d from a massive star that emits ionizing photons at the rate $\mathcal{N}_*$. Let $F_\circ \equiv \mathcal{N}_*/(4\,\pi\,d^2)$ be the ionizing photon flux incident on the globule, and F_1 the flux actually reaching the ionization front at the base of the wind. Define the flux ratio $q \equiv F_\circ/F_1$. Finally, define a nondimensional photoevaporation parameter β as

$$\beta \equiv \frac{\alpha'_{\rm rec}\, F_\circ\, R}{u_1^2} \,,$$

where u_1, the speed at the wind base is approximately equal to a_1, the sound speed in the ionized gas. A high β-value means that a hydrogen atom can recombine many times as it traverses the distance R.

(a) By suitable generalization of equation (15.54), derive an equation for q as a function of β.

(b) Find both β and q numerically for the case $R = 0.2$ pc, $\mathcal{N}_* = 10^{49}$ s^{-1}, and $d = 1$ pc. You may assume $u_1 = 10$ km s^{-1}.

(c) What is n_1, the density at the wind base, under these circumstances?

(d) What is $n_\circ$, the cloud density immediately in front of the ionization front? (*Hint:* Assume that the thermal pressure in the cloud, which has a temperature of 20 K, equals the sum of thermal plus ram pressures in the wind.)

4 - Because of the flux sensitivity of telescopes, any survey of a young cluster can only detect pre-main-sequence stars down to a certain minimum mass. Consider, for example, a 3 Myr old cluster. If the observational sensitivity limit is 0.1 $L_\odot$, what is the lowest-mass star with exactly that age that can be seen? You may proceed as follows:

(a) Assuming the star in question to be fully convective, use equation (16.33) to find the stellar radius R_* as a function of time. You may utilize the simplified version of the equation in the limit that $\tau \gg 1$.

(b) Combine your answer with the blackbody relation, equation (16.9c), to find L_* as a function of $T_{\rm eff}$, M_*, and t.

(c) Finally, use your answer to Problem (16.2) to express $T_{\rm eff}$ as a function of M_*. Hence, find L_* as a function only of M_* and t. From this function, you may derive the desired minimum mass.

(d) Of course, stars in a cluster are actually born over a range of time. Can a star of the minimum mass you obtained be seen if it is just now appearing as an optically visible, pre-main-sequence object?

5 - The luminosity of fully convective, pre-main-sequence stars is controlled by surface conditions, such as the photospheric opacity. On the other hand, the luminosity of radiative stars depends on their internal entropy gradient. It is therefore not surprising that the time to approach the ZAMS, as measured by $t_{\rm KH}$, has a different mass-sensitivity in the two cases.

(a) Begin with convective stars. Using scaling arguments, find the dependence of $t_{\rm KH}$ on M_*. Assume that $T_{\rm eff}$ varies as M_*^n, where $n \approx 0.2$. Assume also that all stars arrive at the ZAMS with identical central temperatures.

(b) Next consider stars that approach the ZAMS along radiative tracks. Again, use scaling arguments to find how $t_{\rm KH}$ varies with M_*. Assume that the internal opacity follows Kramers law, and that the final, central temperature is the same for all masses.

(c) Any protostar that contracts faster than it accretes gas from a surrounding cloud has no pre-main-sequence phase. In such objects, $t_{\rm KH} < t_{\rm acc} \equiv M_*/\dot{M}$. Use your result from (b) to predict the critical transitional mass. Assume $\dot{M} = 1 \times 10^{-5}\ M_\odot\ {\rm yr}^{-1}$ and use the known value of $t_{\rm KH}$ for the Sun. How does your simple estimate compare with detailed numerical results, as exemplified by Figure (16.2)?

6 - Let us see more quantitatively why the central temperature in a contracting brown dwarf first rises, then peaks and falls. Along the way, we will obtain a simple estimate for the maximum mass of such objects.

(a) Since brown dwarfs are fully convective, their internal structure is that of an $n = 3/2$ polytrope. From the analysis in §16.2.3, derive the central pressure P_c as a numerical coefficient β times a function of G, M_*, and the central density ρ_c. Evaluate β numerically.

Your equation gives the central pressure required to resist self-gravity. In a brown dwarf, this pressure arises from a partially degenerate gas. Here, we simplify by supposing P_c to have two components: that of an ideal gas at the appropriate ρ_c and T_c, and the pressure $P_{\rm deg}$ from a *fully* degenerate electron gas, as given in the second form of equation (16.54).

(b) Given these assumptions, find an expression for T_c as a function of ρ_c. For simplicity, set $\mu = 1$ in the ideal gas pressure. Sketch the function $T_c(\rho_c)$. You cannot truly plot the function, since your expression contains the mass M_*, as yet unspecified.

(c) Find $T_{\rm max}$, the maximum central temperature attained by the brown dwarf, for any mass M_*.

(d) What is the largest mass a brown dwarf can have? To find the answer numerically, use the fact that the lowest temperature at which protons can fuse is about 2×10^6 K. How does your answer compare with the correct result, $0.08\ M_\odot$?