# Star Formation: Problem set 6

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## Problem 1

a. The equation of mass continuity is given by formula (3.7) in the book

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial r}(\rho u)\,, \tag{1}$$

where $\rho$ is the mass density and $u$ the velocity. This equation implies that across the ionization front the following is required

$$\rho_0 u_0 = \rho_1 u_1\,, \tag{2}$$

where the subscript 0 refers to the neutral gas approaching the front and the subscript 1 refers to the ionized gas.
In the enviroment of the shock front gravity can be neglected and the equation of momentum conservation becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}\,, \tag{3}$$

I can rewrite this equation by multiplying both sides by $\rho$ and adding an additional factor of $u\partial\rho/\partial t$

$$\begin{aligned} \rho\frac{\partial u}{\partial t} + u\rho\frac{\partial u}{\partial r} + u\frac{\partial \rho}{\partial t} &= -\frac{\partial P}{\partial r} + u\frac{\partial \rho}{\partial t} \\ \frac{\partial(\rho u)}{\partial t} + u\rho\frac{\partial u}{\partial r} &= -\frac{\partial P}{\partial r} + u\frac{\partial \rho}{\partial t}\,, \end{aligned} \tag{4}$$

where in the second step I combined the first and third term on the left-hand side of the first equation.
I can rewrite the second part on the lefthand side to

$$u\rho\frac{\partial u}{\partial r} = \frac{1}{2}\rho\frac{\partial u^2}{\partial r}\,. \tag{5}$$

Furthermore, looking at the eq. 1 I can write the second part of the right-hand side to

$$
\begin{aligned}
u\frac{\partial\rho}{\partial t} &= -u\frac{\partial(\rho u)}{\partial r} \\
&= -u\rho\frac{\partial u}{\partial r} - u^2\frac{\partial\rho}{\partial r} \\
&= -\frac{1}{2}\rho\frac{\partial u^2}{\partial r} - u^2\frac{\partial\rho}{\partial r}\,. 
\end{aligned} \tag{6}
$$

Substituting this back into eq. 4 gives me

$$
\frac{\partial(\rho u)}{\partial t} + \frac{1}{2}\rho\frac{\partial u^2}{\partial r} = \frac{\partial P}{\partial r} - \frac{1}{2}\rho\frac{\partial u^2}{\partial r} - u^2\frac{\partial\rho}{\partial r} \tag{7}
$$

Rearranging the terms

$$
\begin{aligned}
\frac{\partial(\rho u)}{\partial t} &= \frac{\partial P}{\partial r} - \rho\frac{\partial u^2}{\partial r} - u^2\frac{\partial\rho}{\partial r} \\
&= \frac{\partial}{\partial r}(P + \rho u^2)\,. 
\end{aligned} \tag{8}
$$

Now it is clear that momentum conservation across the shock wave requires

$$
P_0 + \rho_0 u_0^2 = P_1 + \rho_1 u_1^2\,. \tag{9}
$$

I can replace the pressure $P$ by $a^2\rho$, where $a^2$ is the isothermal sound speed.

$$
\rho_0\left(a_0^2 + u_0^2\right) = \rho_1\left(a_1^2 + u_1^2\right)\,. \tag{10}
$$

Dividing both sides by $\rho_0$ and using mass conservation so that $\rho_1/\rho_0 = u_0/u_1$ changes the above equation to

$$
\frac{u_0}{u_1}\left(a_1^2 + u_1^2\right) = \left(a_0^2 + u_0^2\right)\,. \tag{11}
$$

Multiplying both sides by $u_0/u_1$

$$
\begin{aligned}
\left(\frac{u_0}{u_1}\right)^2\left(a_1^2 + u_1^2\right) &= \frac{u_0}{u_1}\left(a_0^2 + u_0^2\right) \rightarrow \\
a_1^2\left(\frac{u_0}{u_1}\right)^2 - \frac{u_0}{u_1}\left(a_0^2 + u_0^2\right) + u_0^2 &= 0\,,
\end{aligned} \tag{12}
$$

which is a quadratic equation for $u_0/u_1$.

b. The solution of eq. 12 is

$$
\frac{u_0}{u_1} = \frac{1}{2a_1^2}\left[\left(a_0^2 + u_0^2\right)^2 \pm \sqrt{\left(a_0^2 + u_0^2\right) - 4a_1^2 u_0^2}\right]\,. \tag{13}
$$

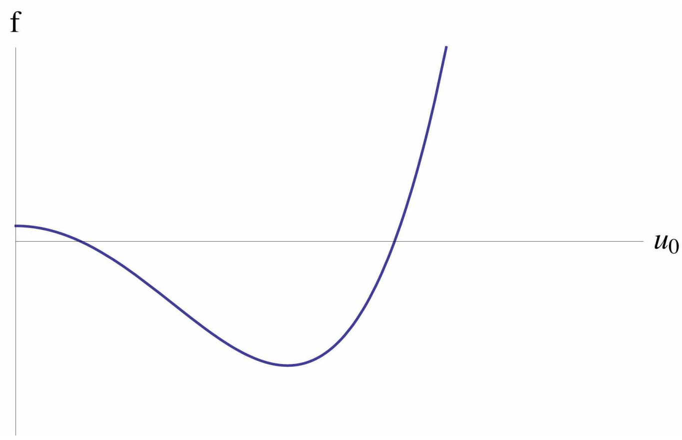


Figure 1: The value of $f$ in eq. 14 as a function of the velocity of the neutral gas $u_0$.

However, for this solution to be physical the formula inside the square root may not become negative. By expanding the formula inside the square root

$$f \equiv \left(a_0^2 + u_0^2\right) - 4a_1^2 u_0^2 = 0 \rightarrow$$
$$u_0^4 - 2\left(2a_1^2 - a_0^2\right) u_0^2 + a_0^4 = 0\,, \tag{14}$$

it becomes apparent that this equation can become negative for intermediate values of $u_0$ (assuming that $a_1^2 \gg a_0^2$) see Figure 1. Thus $u_0^2$ has to be smaller than the smallest root of this function or bigger that the largest root of this function to guarantee that the above equation is positive. The roots of the above equation are given by

$$\begin{aligned} u_0^2 &= \left(2a_1^2 - a_0^2\right) \pm \sqrt{\left(2a_1^2 - a_0^2\right)^2 - a_0^4} \\ &= \left(2a_1^2 - a_0^2\right) \pm 2a_1\sqrt{a_1^2 - a_0^2} \\ &= \left[a_1 \pm \sqrt{a_1^2 - a_0^2}\right]^2 . \end{aligned} \tag{15}$$

Now I see that either

$$u_0 \leq u_- = a_1 - \sqrt{a_1^2 - a_0^2}\,, \tag{16}$$

or

$$u_0 \geq u_+ = a_1 + \sqrt{a_1^2 - a_0^2}\,. \tag{17}$$

Furthermore,

$$u_-u_+ = \left(a_1 - \sqrt{a_1^2 - a_0^2}\right)\left(a_1 + \sqrt{a_1^2 - a_0^2}\right) = a_1^2 - \left(a_1^2 - a_0^2\right) = a_0^2\,. \tag{18}$$

c. For $a_1 \gg a_0$ I can neglect the $a_0^2$ term in the square root of $u_+$ so that

$$u_+ \approx 2a_1\,. \tag{19}$$

d. In an isothermal shock the two isothermal sound speeds, $a$ are equal, thus $a_* = a_0$. Using this, the shock jump condistions in eq. 2 and eq. 10 become

$$\rho_* u_* = \rho_0 u_0 \tag{20}$$

$$\rho_*\left(a_0^2 + u_*^2\right) = \rho_0\left(a_0^2 + u_0^2\right)\,. \tag{21}$$

Using the first of these equations I can substitute $\rho_0 u_0/u_*$ for $\rho_*$ and then divide by $\rho_0$

$$\frac{u_0}{u_*}\left(a_0^2 + u_*^2\right) = a_0^2 + u_0^2\,. \tag{22}$$

Now I can write $a_0$ in terms of $u_0$ and $u_*$

$$\begin{aligned} a_0^2\left(\frac{u_0}{u_*} - 1\right) &= u_0^2 - u_0 u_* \\ a_0^2 &= \frac{u_0^2 u_* - u_0 u_*^2}{u_0 - u_*} \\ a_0^2 &= u_0 u_*\left(\frac{u_0 - u_*}{u_0 - u_*}\right) \\ a_0^2 &= u_0 u_*\,. \end{aligned} \tag{23}$$

Using this equation I can now say that $u_* = a_0^2/u_0$. The shock first appeared when $u_0$ had fallen to $u_+$, which according to eq. 19 is equal to $2a_1$, thus

$$u_* = \frac{a_0^2}{u_0} = \frac{a_0^2}{2a_1}\,. \tag{24}$$

Looking at eq. 16 and assuming that $a_1 \gg a_0$ this becomes

$$\begin{aligned} u_- &= a_1 - a_1\sqrt{1 - \frac{a_0^2}{a_1^2}} \\ &\approx a_1 - a_1\left(1 - \frac{1}{2}\frac{a_0^2}{a_1^2}\right) \\ &\approx \frac{a_0^2}{2a_1}\,, \end{aligned} \tag{25}$$

where in the second step I used the fact that for $x \ll 1$, $\sqrt{1-x} \approx 1 - x/2$. Comparing eq. 24 and 25 I see that $u_* = u_-$, thus the velocity is not in the 'forbidden' regime that was equated in part (b) and thus the inequalties from part (b) are still satisfied.

## Problem 2

a. The momentum equation is given by formula (13.11) in the book, however, there is an extra factor due to the (flux) force of many lines, $f_{\rm rad}$

$$u\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{a_T^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM_*}{r^2} + f_{\rm rad}\,. \tag{26}$$

We can neglect the first term on right, the thermal pressure term, and replace $f_{\rm rad}$ by

$$f_{\rm rad} = \frac{C}{r^2}\left(r^2 u\frac{\mathrm{d}u}{\mathrm{d}r}\right)^\alpha \equiv \frac{C}{r^2}D^\alpha\,, \tag{27}$$

where $(r^2u)\,\mathrm{d}u/\,\mathrm{d}r$ has been denoted as $D$, since it must be a constant. The dimensional constant $C$ is related to the mass loss rate and the nondimensional $\alpha$, which lies between 0 and 1, depends on the optical thickness of the lines.
Finally, multiplying both sides by $r^2$ changes eq. 26 to

$$r^2u\frac{\mathrm{d}u}{\mathrm{d}r} = -GM_* + CD^\alpha\,. \tag{28}$$

The lefthand side is also equal to $D$, thus the equation becomes

$$D = -GM_* + CD^\alpha\,. \tag{29}$$

b. Eq. 29 can be visualized by seeing it as two separate equations. One of the form

$$y_1 = D\,, \tag{30}$$

and one of the form

$$y_2 = -GM_* + CD^\alpha\,. \tag{31}$$

The first one is just a straight line and since $\alpha$ lies between 0 and 1, the second equation is either a straight line with slope 0 or 1, or has the form of a root, see Figure 2. These two lines can have either zero, one or two intersections, depending on the values of $C$ and $\alpha$.
If you want to find the solution to eq 29 that guarantees a single, unique solution, then you want that the tangent of both the lines where eq. 29 is satisfied to be equal, thus

$$\frac{\mathrm{d}y_1}{\mathrm{d}D} = \frac{\mathrm{d}y_2}{\mathrm{d}D} \tag{32}$$

$$1 = \alpha C D^{\alpha-1}\,. \tag{33}$$

Thus

$$C = \frac{1}{\alpha}D^{1-\alpha}\,. \tag{34}$$

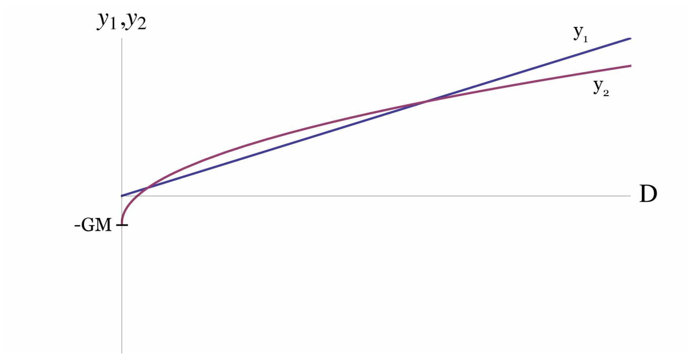


Figure 2: The two lines $y_1$ and $y_2$ as a function of $D$. Here there are two intersections, however, this may change to one or zero intersections for other values of $C$ and $\alpha$.

c. Substituting eq. 34 into eq. 29 gives me

$$\begin{aligned} D &= -GM_* + \frac{1}{\alpha} D \rightarrow \\ (1 - \frac{1}{\alpha})D &= -GM_* \rightarrow \\ D &= \frac{-GM_*}{1-\frac{1}{\alpha}} = \frac{\alpha GM_*}{1-\alpha}. \end{aligned} \tag{35}$$

Remembering that $D = (r^2 u)\,\mathrm{d}u/\,\mathrm{d}r$ I can find a differential equation for $\mathrm{d}u/\,\mathrm{d}r$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{D}{r^2 u} = \frac{\alpha GM_*}{1-\alpha}\frac{1}{r^2 u}. \tag{36}$$

This can be solved:

$$\begin{aligned} \int_{u(R_*)}^{u(r)} u\,\mathrm{d}u &= \frac{\alpha GM_*}{1-\alpha}\int_{R_*}^{r}\frac{1}{r^2}\,\mathrm{d}r \\ \frac{1}{2}\left[u^2\right]_{u(R_*)}^{u(r)} &= -\frac{\alpha GM_*}{1-\alpha}\left[\frac{1}{r}\right]_{R_*}^{r} \\ u(r) &= \sqrt{\frac{2\alpha GM_*}{1-\alpha}\left(\frac{1}{R_*}-\frac{1}{r}\right)}, \end{aligned} \tag{37}$$

here I assumed that $u(R_*) = 0$, where $R_*$ is that star's radius and I have set the integration constant to zero

d. I can find the velocity at infinity by taking the limit of eq. 37 from $r \rightarrow \infty$

$$u_\infty^2 = \frac{2\alpha GM_*}{(1-\alpha)R_*}. \tag{38}$$

The stellar escape speed is given by

$$u_{\text{esc}}^2 = \frac{2GM_*}{R_*} \,. \tag{39}$$

I can substitute this into eq. 38 to find a relation between the escape speed and the terminal speed

$$u_\infty^2 = u_{\text{esc}}^2 \left[\frac{\alpha}{1-\alpha}\right] \,. \tag{40}$$

For a typical value of $\alpha = 0.5$ this becomes

$$u_\infty = u_{\text{esc}} \,. \tag{41}$$

# Problem 3

a. The difference between the situation in this problem and the situation in the book that gives formula (15.54) is that here we do not assume that the stellar flux incident on the globule is entirely absorbed. Therefore, formula (15.54) transforms to

$$F_0 - F_1 = \int_R^\infty n_{\text{gw}}^2 \, \alpha'_{\text{rec}} \, \text{d}r \,, \tag{42}$$

where $F_0 \equiv \mathcal{N}_*/(4\pi d^2)$ is the ionizing photon flux incident on the globule and $F_1$ is the flux actually reaching the ionization front at the base of the wind.
We can still use the righthand side of formula (15.56) for the solution of integral in eq. 42 so that

$$F_0 - F_1 = \frac{(n_1)^2}{3} \alpha'_{\text{rec}} R \,. \tag{43}$$

Defining a nondimensional photoevaporation parameter $\beta$ as

$$\beta \equiv \frac{\alpha'_{\text{rec}} F_0 R}{u_1^2} \,, \tag{44}$$

where $u_1$ is the speed at the wind base, I can write $\alpha'_{\text{rec}} R$ in eq. 43 as $\beta u_1^2 / F_0$ so that eq. 43 becomes

$$F_0 - F_1 = \frac{(n_1 u_1)^2}{3F_0} \beta \,. \tag{45}$$

The flux actually reaching the ionization front is ionizing the still neutral gas beyond the front and is thus creating the wind. Therefore, I can say that $F_1 = n_1 u_1$, where $n_1$ is the density at the wind bas, which changes eq. 45 into

$$F_0 - F_1 = \frac{F_1^2}{3F_0} \beta \,. \tag{46}$$

Dividing both sides by $F_1$ and defining a flux ratio $q \equiv F_0/F_1$ gives me

$$q - 1 = \frac{\beta}{3q} \to q(q-1) = \frac{1}{3}\beta\,. \tag{47}$$

This is a quadratic equation in $q$ and can be solved to give

$$q_{\pm} = \frac{1}{2}\left[1 \pm \sqrt{1 + (4/3)\beta}\right]\,. \tag{48}$$

Since $q > 0$ only the plus sign of these two solutions is the physical one, thus

$$q = \frac{1}{2}\left[1 + \sqrt{1 + (4/3)\beta}\right]\,. \tag{49}$$

b. If $R = 0.2$ pc, $\mathcal{N}_* = 10^{49}\,\mathrm{s}^{-1}$, $d = 1$ pc and $u_1 = 10$ km $\mathrm{s}^{-1}$, then

$$\begin{aligned} F_0 &= \frac{10^{49}\,\mathrm{s}^{-1}}{4\pi(3.086 \cdot 10^{16}\,\mathrm{m})^2} \\ &= 8.36 \cdot 10^{14}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1} \end{aligned}$$

$$\begin{aligned} \beta &= \frac{(2.6 \cdot 10^{-19}\,\mathrm{m}^3\,\mathrm{s}^{-1})(8.36 \cdot 10^{14}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1})(0.2 * 3.086 \cdot 10^{16}\,\mathrm{m})}{(10^4\,\mathrm{m\,s}^{-1})^2} \\ &= 1.34 \cdot 10^4 \end{aligned} \tag{50}$$

$$\begin{aligned} q &= \frac{1}{2}\left[1 + \sqrt{1 + (4/3) * 1.34 \cdot 10^4}\right] \\ &= 67.38 \end{aligned} \tag{51}$$

where for $\alpha'_{\mathrm{rec}}$ I used $2.6 \cdot 10^{-19}\,\mathrm{m}^3\,\mathrm{s}^{-1}$ given by the book on page 520.

c. Since $q = F_0/F_1$ and $F_1 = n_1 u_1$ I can write $n_1$ as

$$n_1 = \frac{F_0}{q u_1} \tag{52}$$

Using the numerical results from part (b) this becomes

$$n_1 = \frac{(8.36 \cdot 10^{14}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1})}{67.38(10^4\,\mathrm{m\,s}^{-1})} = 1.24 \cdot 10^3\,\mathrm{cm}^{-3}\,. \tag{53}$$

d. The cloud density immediately in front of the ionization front, $n_0$ can be found by assuming that the thermal pressure in the cloud equals the sum of the thermal plus ram pressures in the wind. The cloud has a temperature of 20 K.

$$P_{\mathrm{thermal,c}} = P_{\mathrm{thermal,w}} + P_{\mathrm{ram,w}}\,. \tag{54}$$

Using $P = k_B T \rho / \mu m_H$ for the thermal pressure and $P = \rho u^2$ for the ram pressure changes the above equation to

$$\begin{aligned}
\frac{k_B T_c \rho_0}{\mu m_H} &= \frac{k_B T_w \rho_1}{\mu m_H} + \rho_0 u_1^2 \\
a_0^2 \rho_0 &= \rho_1 \left(a_1^2 + u_1^2\right) \\
a_0^2 n_0 &= n_1 \left(a_1^2 + u_1^2\right) \,, \qquad (55)
\end{aligned}$$

where in the second step I used the isothermal sound speed $P/\rho = a^2$. Assuming that $u_1$ is approximately equal to $a_1$ changes the above equation to

$$n_0 = \frac{2 n_1 a_1^2}{a_0^2} \,. \qquad (56)$$

I can calculate $a_0$ by remembering that

$$a_0 = \sqrt{\frac{\mathcal{R} T_c}{\mu}} = 0.29\,\mathrm{km\,s^{-1}} \,. \qquad (57)$$

Using this in eq. 56 gives me a density of

$$\begin{aligned}
n_0 &= \frac{2(1.24 \cdot 10^3\,\mathrm{cm^{-3}})(10\,\mathrm{km\,s^{-1}})^2}{(0.29\,\mathrm{km\,s^{-1}})^2} \\
&= 3 \cdot 10^6\,\mathrm{cm^{-3}} \,. \qquad (58)
\end{aligned}$$

# Problem 4

a. The stellar radius as a function of time for a fully convective star is given by formula (16.33) in the book

$$R_* = R_0 \left(1 + 7\tau\right)^{-1/3} \,, \qquad (59)$$

where $\tau \equiv t/t_0$ and $R_0$ and $t_0$ are the initial (i.e. birthline) values of $R_*$ and $t_{\mathrm{KH}}$, respectively.
In the limit that $\tau \gg 1$ this equation becomes

$$\frac{R_*}{R_0} = \left(7 \frac{t}{t_0}\right)^{-1/3} \,, \qquad (60)$$

b. I can write $t_0$ as

$$t_0 = t_{\mathrm{KH},0} = \frac{G M_*^2}{R_0 L_0} \,, \qquad (61)$$

where $M_0 = M_*$ since there is no more mass accreting onto the star. Also,

$$L_0 = 4\pi R_0^2 \sigma_B T_{\mathrm{eff},0}^4 \,. \qquad (62)$$

However, fully convective stars come down from the birthline onto the ZAMS in an almost vertical line. Therefore, I will assume that $T_{\mathrm{eff},0=T_{\mathrm{eff}}}$. Combining this gives me

$$t_0 = \frac{GM_*^2}{4\pi R_0^3 \sigma_B T_{\mathrm{eff}}^4} \,. \tag{63}$$

Placing this into eq. 60

$$R_* = R_0 \left( \frac{7t(4\pi\sigma_B) R_0^3 T_{\mathrm{eff}}^4}{GM_*^2} \right)^{-1/3} \,. \tag{64}$$

The blackbody relation is given by formula (16.9c) in the book

$$L_* = 4\pi\sigma_B R_*^2 T_{\mathrm{eff}}^4 \,. \tag{65}$$

Now I can substitute eq. 64 into the above equation

$$\begin{aligned} L_* &= 4\pi\sigma_B R_0^2 T_{\mathrm{eff}}^4 \left( \frac{GM_*^2}{7t(4\pi\sigma_B) R_0^3 T_{\mathrm{eff}}^4} \right)^{2/3} \\ &= (4\pi\sigma_B)^{1/3} \left( \frac{GM_*^2}{7t} \right)^{2/3} T_{\mathrm{eff}}^{4/3} \,. \end{aligned} \tag{66}$$

c. The effective temperature has the following dependency on mass

$$T_{\mathrm{eff}} = T_0 \left( \frac{M_*}{M_\odot} \right)^n \,, \tag{67}$$

where $T_0 = 4350$ K and $n \simeq 0.2$. Using this to replace $T_{\mathrm{eff}}$ in eq. 66 gives me

$$L_* = (4\pi\sigma_B)^{1/3} \left( \frac{GM_*^2}{7t} \right)^{2/3} T_0^{4/3} \left( \frac{M_*}{M_\odot} \right)^{4n/3} \,, \tag{68}$$

which now only depends on $M_*$ and $t$. However, I want to know $M_*$ therefore I need to rewrite the above equation.

$$\begin{aligned} M_*^{4(1+n)/3} &= \frac{L_* (7t)^{2/3} M_\odot^{4n/3}}{(4\pi\sigma_B G^2)^{1/3}\; T_0^{4/3}} \\ M_* &= \left[ \frac{L_* (7t)^{2/3} M_\odot^{4n/3}}{(4\pi\sigma_B G^2)^{1/3}\; T_0^{4/3}} \right]^{3/[4(1+n)]} \,. \end{aligned} \tag{69}$$

For an observational sensitivity limit of $0.1 L_\odot$ and age of 3 Myr this mass is

$$\begin{aligned} M_{\mathrm{min}} &= \left[ \frac{(3.846 \cdot 10^{25}\,\mathrm{W})(7 * 9.47 \cdot 10^{13}\,\mathrm{s})^{2/3}}{(4\pi)^{1/3}(6.673 \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})^{2/3}} \times \right. \\ &\quad \left. \frac{(2 \cdot 10^{30}\,\mathrm{kg})^{4/15}}{(5.76 \cdot 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}})^{1/3}(4350\,\mathrm{K})^{4/3}} \right]^{5/8} \\ &= 5.07 \cdot 10^{29}\,\mathrm{kg} \\ &= 0.25\, M_\odot \,. \end{aligned} \tag{70}$$

d. From eq. 69 using that the minimum luminosity that we can observe is a constant, we can deduce that

$$M_{\rm min} \propto t^{5/12} \,. \tag{71}$$

Thus, for younger stars the minimum mass at which we can still detect them is also lower. Therefore, a star of the mass calculated in eq. 70 can indeed be seen if it is a younger star only just appearing as an optically visible, pre-main-sequence object.
Also, looking at Table 16.2 in the book, a star of $0.25\, M_\odot$ has about $1\, L_\odot$. This is higher that the sensitivity limit, thus the star will be visible then.

# Problem 5

a. The Kelvin-Helmholtz timescale is given by

$$t_{\rm KH} = \frac{GM_*^2}{R_* L_*} \,. \tag{72}$$

If all the stars arrive at the ZAMS with identical central temperatures, then formula (16.35) in the book tells us that since $T_c \propto M_*/R_*$ is a constant, it must mean that $M_* \propto R_*$.
The luminosity of the star $L_*$ is given by formula (16.3) in the book

$$L_* = 4\pi R_*^2 \sigma_B T_{\rm eff}^4 \,. \tag{73}$$

I already know that $R_*^2 \propto M_*^2$, assuming also that $T_{\rm eff}$ varies as $M_*^n$, where $n \approx 0.2$, this becomes

$$L_* \propto M_*^2 M_*^{4n} = M_*^{2(1+2n)} \,. \tag{74}$$

With this the $t_{\rm KH}$ becomes

$$t_{\rm KH} \propto \frac{M_*^2}{R_* L_*} \propto \frac{M_*^2}{M_* M_*^{2(1+2n)}} = M_*^{-(1+4n)} \approx M_*^{-1.8} \,. \tag{75}$$

Thus $t_{\rm KH}$ is longer for smaller masses.

b. In radiatively stable regions formula (16.7) can be used to determine $L_*$ if the total mass and radius are used and $(\partial T/\partial M_r)$ is set to $T/M_*$

$$T^3 \frac{T}{M_*} = -\frac{3\kappa L_*}{256\pi^2 \sigma_B R_*^4} \,, \tag{76}$$

where the opacity $\kappa$ follows Kramer's Law, $\kappa \propto \rho T^{-7/2}$.
From formula (11.2a) in the book I see that $T$ scales as $M_* R_*^{-1}$. The density $\rho$ can be replaced by $M_* R_*^{-3}$, so that $\kappa \propto M_*^{-5/2} R_*^{1/2}$. Rearranging the terms in eq. 76

$$L_* \propto \frac{T^4 R_*^4}{\kappa M_*} \propto \frac{M_*^4 R_*^{-4} R_*^4}{M_*^{-5/2} R_*^{1/2} M_*} = M_*^{11/2} R_*^{-1/2} \,. \tag{77}$$

Finally, the final central temperature is the same for all masses. Therefore, I will assume that $T_c \propto M_* R_*^{-1}$ so that $R_* \propto M_*$. This implies that

$$L_* \propto M_*^5 \,, \tag{78}$$

and that $t_{\rm KH}$ becomes

$$t_{\rm KH} \propto \frac{M_*^2}{R_* L_*} \propto \frac{M_*^2}{M_* M_*^5} = M_*^{-4} \,. \tag{79}$$

c. Stars that have no pr-main-sequance have have $t_{\rm KH} < t_{\rm acc} \equiv M_*/\dot{M}$. Assume that $\dot{M} = 1 \cdot 10^{-5}\, M_\odot\, {\rm yr}^{-1}$.
The Kelvin-Helmholtz time of the Sun is

$$\begin{aligned} t_{\rm KH,\odot} &= \frac{G M_\odot}{R_\odot L_\odot} \\ &= \frac{\left(6.673 \cdot 10^{-11}\,{\rm m^3\,kg^{-1}\,s^{-2}}\right)\left(2 \cdot 10^{30}\,{\rm kg}\right)^2}{(6.955 \cdot 10^{8}\,{\rm m})\,(3.846 \cdot 10^{26}\,{\rm W})} \\ &= 9.98 \cdot 10^{14}\,{\rm s} \\ &= 3.16 \cdot 10^{7}\,{\rm yr}\,. \end{aligned} \tag{80}$$

From part (b) I know that

$$\frac{t_{\rm KH}}{t_{\rm KH,\odot}} = \left(\frac{M_\odot}{M_*}\right)^4 \,. \tag{81}$$

Using this in the inequality

$$\begin{aligned} t_{\rm KH} &< t_{\rm acc} \equiv \frac{M_*}{\dot{M}} \\ t_{\rm KH,\odot}\left(\frac{M_\odot}{M_*}\right)^4 &< \frac{M_*}{\dot{M}} \\ t_{\rm KH,\odot}\dot{M} M_\odot^4 &< M_*^5 \,. \end{aligned} \tag{82}$$

Now I have found $M_*$ in terms of which I know the value

$$\begin{aligned} M_* &> \left(t_{\rm KH,\odot}\dot{M} M_\odot^4\right)^{1/5} \\ M_* &> \left[(3.16 \cdot 10^{7}\,{\rm yr})(1 \cdot 10^{-5}\,M_\odot\,{\rm yr}^{-1}) M_\odot^4\right]^{1/5} \\ M_* &> 3.16\,M_\odot\,. \end{aligned} \tag{83}$$

The detailed numerical results displayed in Figure (16.2) in the book show that there is no pre-main-sequence phase, thus $t_{\rm KH} < t_{\rm acc}$, for $M_* \gtrsim 6\,M_\odot$. The simple estimate calculated here is only off by a factor of two, which seems like a good result considering the simple scaling arguments used.

## Problem 6

a. A fully convective star is a $n = 3/2$ polytrope. The relation between the pressure and density is then given by formula (16.15) in the book

$$P = K\rho^{5/3} \,. \tag{84}$$

According to formula's (16.21) and (16.22) in the book the radius and mass of a star of this type are given by

$$\begin{aligned} R_* &= a\xi_0 = \left[\frac{5K}{8\pi G\rho_c^{1/3}}\right]^{1/2} \xi_0 \,, & (85) \\ M_* &= 4\pi a^3 \rho_c \left(-\xi^2 \frac{\partial\theta}{\partial\xi}\right)_0 = 4\pi \left[\frac{5K}{8\pi G}\right]^{3/2} \rho_c^{1/2} \left(-\xi^2 \frac{\partial\theta}{\partial\xi}\right)_0 . & (86) \end{aligned}$$

I can rewrite eq. 85 so I can eliminate $\rho_c$ between the radius and the mass

$$\rho_c^{1/2} = \left[\frac{5K}{8\pi G}\right]^{3/2} \left(\frac{\xi_0}{R_*}\right)^3 . \tag{87}$$

Substituting this into eq. 86

$$M_* = 4\pi \left[\frac{5K}{8\pi G}\right]^3 \left(\frac{\xi_0}{R_*}\right)^3 \left(-\xi^2 \frac{\partial\theta}{\partial\xi}\right)_0 . \tag{88}$$

Now I can rewrite this to give my $K$

$$K = \frac{2}{5} \frac{(4\pi)^{2/3}}{\left[\xi_0^3 \left(-\xi^2 \frac{\partial\theta}{\partial\xi}\right)_0\right]^{1/3}} G M_*^{1/3} R_* \,. \tag{89}$$

However, there is still a dependence on radius. This terms can be eliminated by remembering that the average densty inside a star can be written as

$$\bar{\rho} = \frac{3M_*}{4\pi R_*^3} \,. \tag{90}$$

I can replace the mass of the star by eq. 86 and the radius by eq. 85

$$\bar{\rho} = \frac{-12\pi a^3 \rho_c \xi_0^2 \left(\frac{\partial\theta}{\partial\xi}\right)_0}{4\pi a^3 \xi_0^3} = -\frac{3}{\xi_0} \rho_c \left(\frac{\partial\theta}{\partial\xi}\right)_0 . \tag{91}$$

Using this, I can rewrite eq. 90 to the radius

$$R_* = \left(\frac{3M_*}{4\pi\bar{\rho}}\right)^{1/3} = \left(-\frac{\xi_0 M_*}{4\pi\rho_c \left(\frac{\partial\theta}{\partial\xi}\right)_0}\right)^{1/3} , \tag{92}$$

Now, I can substitute this into eq. 89 so $K$ is a function only of $M_*$ and $\rho_c$

$$K = \frac{2}{5} \frac{(4\pi)^{1/3}}{\left[\left(-\xi^2 \frac{\partial\theta}{\partial\xi}\right)_0\right]^{2/3}} G M_*^{2/3} \rho_c^{-1/3} \,. \tag{93}$$

Thus the central density is

$$\begin{aligned} P_c &= K\rho_c^{5/3} = \frac{2}{5} \frac{(4\pi)^{1/3}}{\left[\left(-\xi^2 \frac{\partial\theta}{\partial\xi}\right)_0\right]^{2/3}} G M_*^{2/3} \rho_c^{4/3} \,, \\ &= \beta G M_*^{2/3} \rho_c^{4/3} \,. \end{aligned} \tag{94}$$

Here

$$\beta = \frac{2}{5} \frac{(4\pi)^{1/3}}{\left[\left(-\xi^2 \frac{\partial\theta}{\partial\xi}\right)_0\right]^{2/3}} = 0.478 \,, \tag{95}$$

where

$$\left(-\xi^2 \frac{\partial\theta}{\partial\xi}\right)_0 = 2.71406 \tag{96}$$

b. Assume that $P_c$ is comprised of two components; one following the ideal gas law and one for a fully degenerate gas where the pressure is given by formula (16.54) in the book, so that

$$\begin{aligned} P_c &= P_{\text{ideal}} + P_{\text{deg}} \,, \\ &= \frac{k_b}{m_H} \rho_c T_c + K_{\text{deg}}\, \rho_c^{5/3} \,, \end{aligned} \tag{97}$$

where $\mu$ has been set to one in the ideal gas law and $K_{\text{deg}} = 7.7 \cdot 10^{12}$ [$\text{g}^{-2/3}$ $\text{cm}^4$ $\text{s}^{-2}$]. This can be rewritten to give an expression for $T_c$ as a function of $\rho_c$ (and $M_*$)

$$\begin{aligned} T_c &= \frac{m_H}{k_b \rho_c} \left(P_c - K_{\text{deg}}\, \rho_c^{5/3}\right) \\ &= \frac{m_H}{k_b} \left(\beta G M_*^{2/3} \rho_c^{1/3} - K_{\text{deg}}\, \rho_c^{2/3}\right) \,, \end{aligned} \tag{98}$$

where eq. 94 was used for $P_c$. For small $\rho_c$ the first term is dominant and $T_c$ will rise as $\rho_c^{1/3}$, while for large $\rho_c$ the second term becomes dominant and the central temperature starts to decline as $\rho_c^{2/3}$ until $T_c = 0.$. The resulting plot can be found in Figure 3. Since the mass is still unspecified, there are no numbers given on the axes.

c. The maximum central temperature, can be found by differentiating eq. 98 with respect to $\rho_c$ and setting this equal to zero

$$\frac{\partial T_c}{\partial \rho_c} = \frac{m_H}{k_b} \left(\frac{1}{3} \beta G M_*^{2/3} \rho_c^{-2/3} - \frac{2}{3} K_{\text{deg}}\, \rho_c^{-1/3}\right) = 0 \,. \tag{99}$$

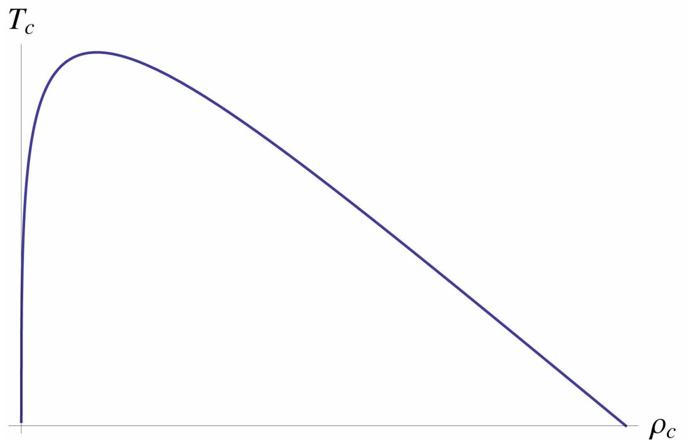


Figure 3: The central temperature, $T_c$, as a function of the central density $\rho_c$

This implies that

$$\begin{aligned}\beta G M_*^{2/3}\rho_c^{-2/3} &= 2K_{\rm deg}\,\rho_c^{-1/3} \\ \rho_{c,\max} &= \left[\frac{\beta G M_*^{2/3}}{2K_{\rm deg}}\right]^3 . \end{aligned} \tag{100}$$

I can now replace this density in eq. 98 to find $T_{\max}$ for any mass

$$\begin{aligned} T_{\max} &= \frac{m_H}{k_b}\left(\beta G M_*^{2/3}\left[\frac{\beta G M_*^{2/3}}{2K_{\rm deg}}\right] - K_{\rm deg}\left[\frac{\beta G M_*^{2/3}}{2K_{\rm deg}}\right]^2\right) \\ &= \frac{m_H}{k_b}\left[\frac{\beta G M_*^{2/3}}{K_{\rm deg}}\right]^2\left(\frac{1}{2}K_{\rm deg} - \frac{1}{4}K_{\rm deg}\right) \\ &= \frac{m_H}{4k_b}\frac{\left(\beta G M_*^{2/3}\right)^2}{K_{\rm deg}} . \end{aligned} \tag{101}$$

d. The largest mass a brown dwarf can have is set by the value of the maximum temperature that is still below the hydrogen burning temperature.. The lowest temperature at which protons can fus is about $2\cdot 10^6$ K. Thus the mass corresponding to this $T_{\max}$ is

$$M_* = \left[\frac{4k_b T_{\max} K_{\rm deg}}{m_H \beta^2 G^2}\right]^{3/4} \tag{102}$$

$$\begin{aligned} &= \left[\frac{4(1.38\cdot 10^{-16}\,{\rm erg\,K^{-1}})(2\cdot 10^6\,{\rm K})(7.7\cdot 10^{12}\,{\rm g^{-2/3}\,cm^4\,s^{-2}})}{(1.67\cdot 10^{-24}\,{\rm g})(0.478^2)(6.673\cdot 10^{-8}\,{\rm cm^3\,g^{-1}\,s^{-2}})^2}\right]^{3/4} \\ &= 1.06\cdot 10^{32}\,{\rm g} = 0.05 M_\odot , \end{aligned} \tag{103}$$

which comes very close to the correct result of $0.08M_\odot$.