AY250 Assignment 7: Solutions

Problem 1

a)

The accretion luminosity is

$$L_{\text{acc}} = \frac{GM_* \dot{M}}{R_*}$$

Here, L_{acc} is the continuum excess value of 0.18 $L_\odot$ and M_* is, according to figure 1.18 in the book, about 0.8 $M_\odot$. R_* can be calculated from $L_* = 4\pi\sigma_B R_*^2 T_{\text{eff}}^4$ because we know the stellar luminosity and effective temperature; the answer turns about to be $R_* =$ ~~2.7~~ 2.1 $R_\odot$. Putting all that together, I find that

$$\dot{M} = \frac{R_* L_{\text{accr}}}{GM_*} \approx 1.9 \cdot 10^{-8}\ M_\odot\ \text{yr}^{-1}$$

b)

Provided the gas has been falling over a distance of several stellar radii, its velocity should be close to free fall. Using

$$V = \sqrt{\frac{2GM_*}{R_*}}$$

I get that $V \approx$ ~~340~~ 380 km/s. Referring to figure 17.12, I see that the absorption dips are more consistent with infall of about 250 km/s, but our answer is good to within a factor of two.

c)

The excess luminosity can be written

$$L_{\text{excess}} = 0.18 L_\odot = f 4\pi\sigma_B R_*^2 T_{\text{con}}^4 \; .$$

Solving that equation for f, I find

$$f \approx 3 \cdot 10^{-4}$$

2×10^{-3}

Problem 2

a)

Imagine you are sitting on the disk facing the star. Consider a segment of this circle located somewhat off center. If you construct a Cartesian coordinate system centered on the segment so that the x-axis points straight down (parallel to the normal vector to the disk) and the z-axis points away from the center of the star, you should get something like figure 3a below. For a right-handed system, the y-axis will then point to your left.

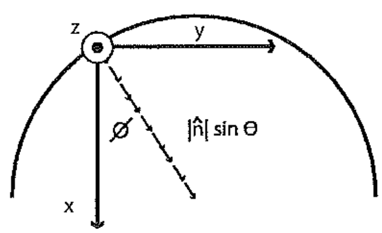

Figure 3a

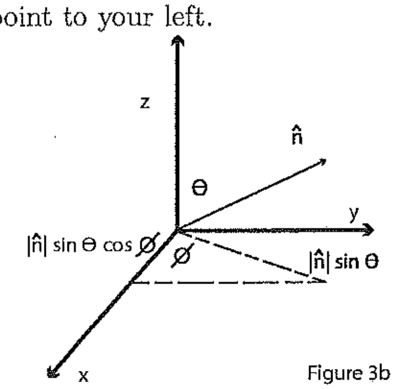

Figure 3b

Now, consider the vector $\hat{n}$ that points from the annulus to you. This vector always forms an angle of θ with the z-axis. Furthermore, the projection of this vector onto the xy plane can form an angle of anywhere between -90 and 90 with the x-axis. Call that angle ϕ. We would like to find the projection of those rays onto the x-axis. To find this, refer to figure 3b. First we project the ray onto the xy-plane, forming a vector of length $|\hat{n}| \sin\theta$ that forms an angle of ϕ with the x-axis. The projection onto the vertical is then $\cos\phi \sin\theta$. To find the average over the entire half-annulus, we integrate from $\phi = -\pi/2$ to $\pi/2$ and divide by π:

$$f_\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos\phi \sin\theta \, d\theta = \frac{2}{\pi}\sin\theta \; .$$

b)

The luminosity is twice the integral of $F_{\rm rad}$, since both sides of the disk are illuminated. Using equation 17.18,

$$\begin{aligned}
L_D &= 2\int_{R_0}^{R_f} F_{\rm rad} 2\pi R dR \\
&= 4\sigma_B T_{\rm eff}^4 \int_{R_*}^{R_f} R\left(\arcsin\left(\frac{R_*}{R}\right) - \frac{R_*}{R}\sqrt{1-\left(\frac{R_*}{R}\right)^2}\right) dR \\
&= 4\sigma_B R_*^2 T_{\rm eff}^4 \int_1^{R_D/R_*} \left(u \arcsin\left(\frac{1}{u}\right) - \sqrt{1-\left(\frac{1}{u}\right)^2}\right) dR
\end{aligned}$$

Where in the last step I've defined $u = R/R_*$. The first term can be integrated by parts. The result is

$$\int u \arcsin\left(\frac{1}{u}\right) du = \frac{1}{2}u^2 \arcsin\left(\frac{1}{u}\right) + \frac{1}{2}\sqrt{u^2-1}$$

The second term is equivalent to

$$\int \frac{\sqrt{u^2-1}}{u} du$$

which can be looked up or done on Mathematica. The result is

$$\int \frac{\sqrt{u^2-1}}{u} du = \sqrt{u^2-1} - \arccos\left(\frac{1}{u}\right)$$

Putting that all together, you should get

$$L_D = 4R_*^2 \sigma_B T_{\rm eff}^4 \left(\frac{1}{2}u_D^2 \arcsin\left(\frac{1}{u_D}\right) - \frac{1}{2}\sqrt{u_D^2 - 1} + \arccos\left(\frac{1}{u_D}\right) - \frac{\pi}{4}\right)$$

where $u_D = R_D/R_*$.

c)

Now, for large u_D, $\arcsin\left(\frac{1}{u_D}\right) \to \frac{1}{u_D}$ and $\arccos\left(\frac{1}{u_D}\right) \to \frac{\pi}{2}$, so we're left with

$$L_D = 4R_*^2\sigma_B T_{\rm eff}^4\left(\frac{u_D}{2} - \frac{u_D}{2} + \frac{\pi}{2} - \frac{\pi}{4}\right) = \pi R_*^2\sigma_B T_{\rm eff}^4$$

Or, in other words, precisely $\frac{1}{4}L_*$. This is slightly larger than the approximate value of 0.2 quoted in the text.

Problem 3

a)

Vertical hydrostatic equilibrium is expressed by the unnumbered equation before (11.34):

$$-\frac{\partial P}{\partial z} = \frac{-\rho\, G\, M_*\, z}{\varpi^3}.$$

Since the gas is isothermal, $P = \rho\, a_T^2$. Thus

$$-\frac{\partial \rho}{\partial z} = \frac{-\rho\, G\, M_* z}{a_T^2\, \varpi^3}\ ,$$

which has the solution

$$\rho(z) = \rho_c \exp\left[-\left(\frac{z}{\Delta z}\right)^2\right]\ .$$

Here, ρ_c is the midplane density. The scale height Δz is given by

$$\Delta z \equiv \frac{a_T}{\Omega_{\rm Kep}}\ ,$$

where the Keplerian angular rotation rate is

$$\Omega_{\rm Kep} \equiv \left(\frac{G\, M_*}{\varpi^3}\right)^{1/2}\ .$$

Thus the total surface density $\Sigma = 2\int_0^\infty \rho\, dz$ is

$$\Sigma = \sqrt{\pi}\, \rho_c\, \Delta z\ .$$

after integrating over the Gaussian density profile. Supplying the expression for Δz and applying isothermality, we solve for the midplane pressure:

$$P_c = \frac{a_T\, \Omega_{\rm Kep}\, \Sigma}{\sqrt{\pi}}\ .$$

b)

To find α, we note the proportionalities:

$$\begin{aligned} a_T &\propto T^{1/2} \propto \varpi^{-q/2} \,, \\ \Omega_{\rm Kep} &\propto \varpi^{-3/2} \,, \\ \Sigma &\propto \varpi^{-n} \,. \end{aligned}$$

Thus, from (a), we also have

$$P_c \propto \varpi^{-n-q/2-3/2} \,,$$

so that

$$\alpha = -n - \frac{q}{2} - \frac{3}{2} \,.$$

c)

At the midplane, centrifugal balance of the gas implies

$$\frac{\rho_c\, u_\phi^2}{\varpi} = \frac{\rho_c\, G\, M_*}{\varpi^2} + \frac{\partial P_c}{\partial \varpi} \,.$$

We may solve for u_ϕ^2 in terms of the Keplerian speed $V_{\rm Kep}$:

$$\begin{aligned} u_\phi^2 &= \frac{G\, M_*}{\varpi} + \frac{\varpi\, \partial P_c}{\rho_c\, \partial \varpi} \,, \\ &= V_{\rm Kep}^2 + \alpha\, a_T^2 \,. \end{aligned}$$

Note that $\alpha < 0$, so that $u_\phi < V_{\rm Kep}$. If we manipulate the above equation and use the fact that u_ϕ and $V_{\rm Kep}$ are close in magnitude, we find

$$2\, V_{\rm Kep}\, (u_\phi - V_{\rm Kep}) = \alpha\, a_T^2 \,,$$

so that

$$\frac{\Delta V}{V_{\rm Kep}} = \frac{|\alpha|\, a_T^2}{2\, V_{\rm Kep}^2} \,.$$

d)

Numerically, we have

$$\begin{aligned}
\alpha &= -\frac{3}{2} - \frac{1}{4} - \frac{3}{2} = -\frac{13}{4} \\
a_T &= \sqrt{\frac{\mathcal{R}T}{\mu}} = 1 \text{ km s}^{-1} \\
V_{\rm Kep} &= \sqrt{\frac{GM_\odot}{\varpi}} = 30 \text{ km s}^{-1} .
\end{aligned}$$

In evaluating a_T, μ has been set equal to 2. We find that

$$\frac{\Delta V}{V_{\rm Kep}} = 2 \times 10^{-3} .$$

Problem 4

Part a

As derived in the solution to a previous problem, the factor by which gravitational focusing enhances the cross-section is

$$1 + \frac{2GM}{Rv^2} = 1 + 2\theta$$

where θ is the Safronov number. So the collisional cross-section is

$$\sigma = \pi R^2(1 + 2\theta)$$

The rate of collisions is given by the now familiar formula $n\sigma v$, where n is the number density of the planetesimals. This means that the rate at which mass is accreting onto the large planetesimal is

$$\dot{M} = \rho_s \sigma v = \rho_s[\pi R^2(1 + 2\theta)]v$$

Part b

To find the scale height, we begin with the equation for hydrostatic equilibrium for gas at cylindrical radius ϖ and height z:

$$-\frac{\partial P}{\partial z} = \frac{\rho GMz}{\varpi^3}$$

Separating the differential equation and integrating, and using the isothermal equation of state $P = \rho {a_T}^2$, we obtain

$$\rho(\varpi, z) = \rho_0(\varpi)\exp\left[-\frac{1}{2}\frac{GM}{\varpi}\frac{1}{{a_T}^2}\frac{z^2}{\varpi^2}\right]$$

Now we invoke

$$V_{\text{kep}} = \sqrt{\frac{GM}{\varpi}} \qquad \Omega_{\text{kep}} = \frac{V_{\text{kep}}}{\varpi} \qquad v_{\text{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{3}a_T$$

Substituting these expressions into our expression for the density, we find

$$\rho(\varpi, z) = \rho_0(\varpi)\exp\left[-z^2\left(\frac{3}{2}\frac{{\Omega_{\text{kep}}}^2}{v^2}\right)\right]$$

So the scale height h is given by

$$h = \sqrt{\frac{2}{3}}\frac{v}{\Omega_{\text{kep}}}$$

Part c

Using the definition of Σ and our expression for h, we find

$$\rho_s\sqrt{\frac{2}{3}}\frac{v}{\Omega_{\text{kep}}} = \Sigma \implies \rho_s v = \sqrt{\frac{3}{2}}\Sigma\Omega_{\text{kep}}$$

Substituting this into our equation for $\dot{M}$ we obtain

$$\dot{M} = \sqrt{\frac{3}{2}}\Sigma\Omega_{\text{kep}}[\pi R^2(1 + 2\theta)]$$

Part d

By definition,

$$\rho_p = \frac{M}{4/3\pi R^3} \implies R^3 = \frac{M}{4/3\pi\rho_p}$$

We now take the time derivative of both sides and solve for $\dot{R}$ to get

$$\dot{R} = \frac{\dot{M}}{4\pi\rho_p R^2} = \sqrt{\frac{3}{32}}\frac{\Sigma\Omega_{\text{kep}}}{\rho_p}(1+2\theta)$$

If θ is fixed, then the right-hand side of the above equation is constant, and R grows at a constant rate

Part e

Now if θ is much larger than 1, then we have approximately

$$\dot{R} = \sqrt{\frac{3}{32}}\frac{\Sigma\Omega_{\text{kep}}}{\rho_p}(2\theta) = \sqrt{\frac{3}{32}}\frac{\Sigma\Omega_{\text{kep}}}{\rho_p}\left(\frac{2G}{v^2}\frac{4}{3}\pi\rho_p R^2\right)$$

We can lump all the constants into the symbol K to write

$$\dot{R} = KR^2$$

Now consider two planetesimals with radii $R_1(t)$ and $R_2(t)$ such that $R_1 > R_2$. Then

$$\frac{\dot{R}_1}{\dot{R}_2} = \frac{R_1^2}{R_2^2}$$

We can integrate this equation to yield

$$R_1(t) = \frac{R_2(t)}{1 - AR_2(t)}$$

where

$$A \equiv \frac{1}{R_2(0)} - \frac{1}{R_1(0)}$$

It is now evident that when R_2 rises to a value of $1/A$, R_1 runs away to infinity.

Star Formation: Problem set 7

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Problem 5

a. Assuming that L_{int} is L_{rad} as given in formula (11.29) in the book then

$$L_{\text{int}} = \frac{64\pi G M_r \sigma_B T^3}{3\kappa} \left(\frac{\partial T}{\partial P}\right)_s . \tag{1}$$

If κ follows Kramers' Law then

$$\kappa = \kappa_0 \rho T^{-7/2} . \tag{2}$$

I can replace ρ by $\mu m_H P / k_B T$ so that

$$\kappa = \kappa_0' P T^{-9/2} , \tag{3}$$

where κ_0' incorporates the new constant from replacing ρ. Furthermore, for a monoatomic ideal gas

$$\left(\frac{\partial T}{\partial P}\right)_s = \frac{2T}{5P} , \tag{4}$$

thus this changes L_{int} to

$$L_{\text{int}} = \frac{128\pi G M_* \sigma_B}{15\kappa_0'} \frac{m T^4}{T^{-9/2} P^2} = \frac{128\pi G M_* \sigma_B}{15\kappa_0'} \frac{m T^{17/2}}{P^2} , \tag{5}$$

where I used $m \equiv M_r / M_*$. From formula (11.21) in the book and the fact that the star consists of a monoatomic ideal gas in which the specific entropy is a spatial constant I see that

$$T^{3/2} \propto \rho , \tag{6}$$

thus I can write the pressure as

$$P \propto \rho T \propto T^{5/2} \rightarrow T \propto P^{2/5} . \tag{7}$$

I can replace T in eq. 5 and place all the constants into a dimensional constant L_1

$$L_{\text{int}} = L_1 \frac{m T^{17/2}}{p^2} = L_1 \frac{m p^{17/5}}{p^2} = L_1 m p^{7/5} , \tag{8}$$

where $p \equiv P/P_0$ and P_0 is the central pressure.

b. Starting with the formula (11.14) in the book and multiplying both sides by r^2/ρ and taking the derivative

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -G\frac{\mathrm{d}M_r}{\mathrm{d}r}\,. \tag{9}$$

I can replace $\mathrm{d}M_r/\,\mathrm{d}r$ by formula (10.26) in the book

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2\rho\,. \tag{10}$$

So that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G r^2\rho\,. \tag{11}$$

Finally, dividing both sides by r^2 gives me

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho\,. \tag{12}$$

Now I can replace ρ by $\rho_0\phi^{3/2}$, where ρ_0 is the central value. Also, from eq. 6 I can say that

$$P = K'T\rho = K\rho^{5/3} = K\rho_0^{5/3}\phi^{5/2}\,, \tag{13}$$

where K' and K incorporate all the constants. From the above equation also follows that

$$P = P_0\phi^{5/2} \rightarrow P_0 = K\rho_0^{5/3}\,. \tag{14}$$

Thus eq. 12 becomes

$$\begin{aligned} \frac{1}{4\pi G\rho_0 r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{5/2P_0\phi^{3/2}r^2}{\rho_0\phi^{3/2}}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) &= -\phi^{3/2} \\ \frac{5P_0}{8\pi G\rho_0^2 r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) &= -\phi^{3/2}\,. \end{aligned} \tag{15}$$

I can also replace r by $r_0\xi$ where

$$r_0 = \sqrt{\frac{5P_0}{8\pi G\rho_0^2}}\,. \tag{16}$$

This changes eq. 15 into

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) = -\phi^{3/2}\,, \tag{17}$$

which is actually the Lane-Emden equation for a polytrope with $n = 3/2$.

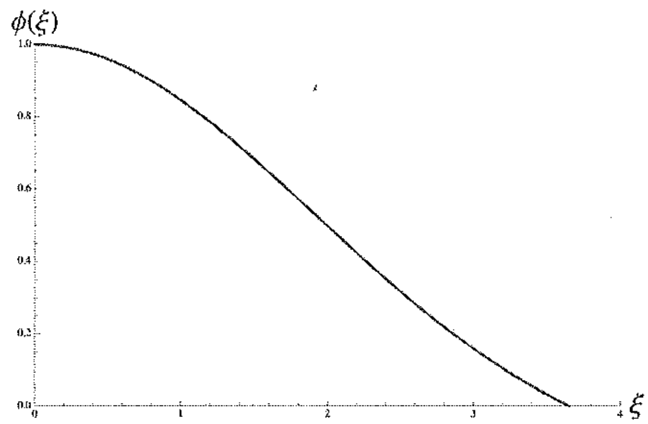

Figure 1: The function $\phi(\xi)$ as a function of ξ.

c Eq. 17 has the following boundary conditions

$$\begin{aligned}\phi(0) &= 1 \\ \phi'(0) &= 0\,.\end{aligned} \tag{18}$$

The first one states that at the center the density should equal ρ_0. The second one comes from the fact that at $r = 0$, $\mathrm{d}P/\,\mathrm{d}r = \rho_0 g(0) = 0$, since there is no mass inside zero radius and $\mathrm{d}P/\,\mathrm{d}r \propto \mathrm{d}\phi/\,\mathrm{d}\xi$. Numerically integrating eq. 17 with these boundary conditions gives a function of $\phi(\xi)$ as plotted in Figure 1

d. I can subsitute M, r and ρ for m, ξ and ϕ in eq. 10

$$\frac{\partial m}{\partial \xi} = \frac{4\pi r_0^3 \rho_0}{M_*} \xi^2 \phi^{3/2}\,. \tag{19}$$

If I integrate this over the total radius of the star, then $m = 1$ thus

$$1 = \frac{4\pi r_0^3 \rho_0}{M_*} \int_0^{\xi_0} \xi^2 \phi^{3/2}\,\mathrm{d}\xi\,, \tag{20}$$

where $R_* = r_0 \xi_0$. Looking at eq. 17 I can replace the part inside the integral by

$$\begin{aligned}1 &= -\frac{4\pi r_0^3 \rho_0}{M_*} \int_0^{\xi_0} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) \mathrm{d}\xi \\ &= -\frac{4\pi r_0^3 \rho_0}{M_*} \left(\xi^2 \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)_{\xi_0}\,.\end{aligned} \tag{21}$$

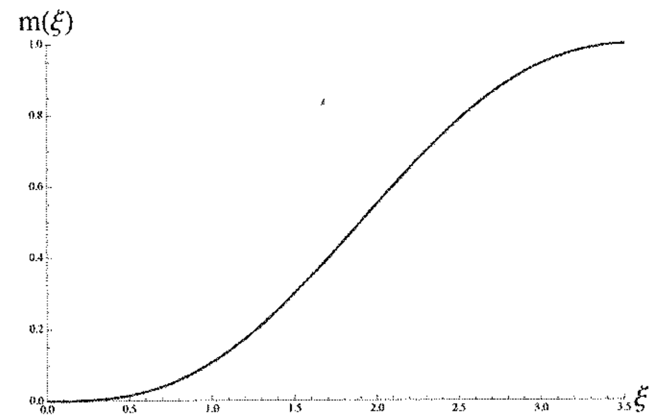

Figure 2: The function $m(\xi)$ as a function of ξ.

From section 16.2.3 in the book the term inside the parentheses, which is evaluated at ξ_0, has the value -2.71 Thus the prefactor has a value of

$$\frac{4\pi r_0^3 \rho_0}{M_*} = \frac{1}{2.71} = 0.37\,. \tag{22}$$

e. I can also choose not to integrate all the way to the radius of the star. This implies that eq. 21 changes into

$$m(\xi) = -0.37\left(\xi^2 \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)\,. \tag{23}$$

This can be evaluated numerically. The corresponding plot can be found in Figure 2

f. From eq. 14 I know that

$$P = P_0 \phi^{5/2}\,. \tag{24}$$

Thus replacing P by $p \equiv P/P_0$ this becomes

$$p = \phi^{5/2}\,. \tag{25}$$

With this eq. 8 changes into

$$\frac{L_{\rm int}}{L_1} = m\phi^{7/2}\,. \tag{26}$$

Now I need to know the function $\phi(m)$. I can find this relation with the use of $\phi(\xi)$ and $m(\xi)$ found in problems (c) and (e). Using numerical techniques to find $\phi(m)$ and using this in eq. 26 I get the internal luminosity distribution as a function of m. The corresponding plot can be found in Figure 3.

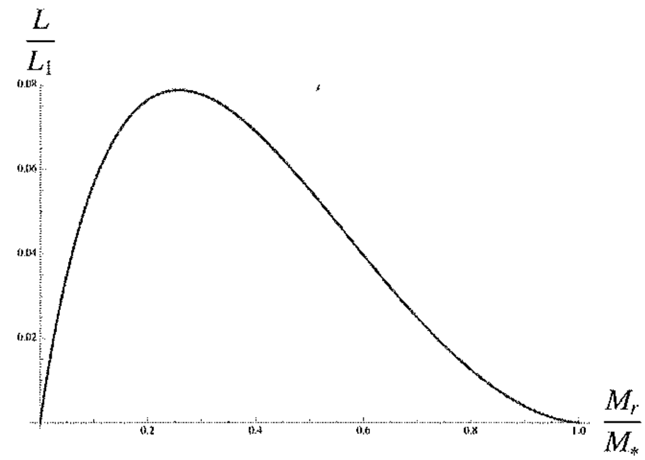

Figure 3: The internal luminosity $L_{\rm int}/L_1$ as a function of mass M_r/M_*.

Problem 6

a. The energy generation rate for the CNO cycle is

$$\epsilon_{\rm CNO} = 1.33 \cdot 10^{27}\,{\rm erg\,g^{-1}\,s^{-1}} \left(\frac{\rho}{100\,{\rm g\,cm^{-3}}}\right) \left(\frac{T}{10^7\,{\rm K}}\right) \exp -\theta\,, \quad (27)$$

where

$$\theta \equiv 70.7 \left(\frac{T}{10^7\,{\rm K}}\right)^{-1/3}\,. \quad (28)$$

Two ZAMS stars of 2 and 5 $M_\odot$ have corresponding central densities of $\log\rho_c = 1.8$ and 1.2, where ρ_c is in g cm^{-3} and central temperatures of $\log T = 7.30$ and 7.42. The CNO energy generation rate at the center from eq. 27 for the 2 $M_\odot$ star is then

$$\begin{aligned} \epsilon_{\rm CNO} &= 1.33 \cdot 10^{27}\,{\rm erg\,g^{-1}\,s^{-1}} \left(\frac{10^{1.8}\,{\rm g\,cm^{-3}}}{100\,{\rm g\,cm^{-3}}}\right) \left(\frac{10^{7.30}\,{\rm K}}{10^7\,{\rm K}}\right) \times \\ &\quad \exp\left[-70.7\left(\frac{10^{7.30}\,{\rm K}}{10^7\,{\rm K}}\right)^{-1/3}\right] \\ &= 2.2 \cdot 10^2\,{\rm erg\,g^{-1}\,s^{-1}}\,. \end{aligned} \quad (29)$$

For the 5 $M_\odot$ this is

$$\begin{aligned}\epsilon_{\rm CNO} &= 1.33\cdot 10^{27}\,{\rm erg\,g^{-1}\,s^{-1}}\left(\frac{10^{1.2}\,{\rm g\,cm^{-3}}}{100\,{\rm g\,cm^{-3}}}\right)\left(\frac{10^{7.42}\,{\rm K}}{10^7\,{\rm K}}\right)\times \\ & \qquad \exp\left[-70.7\left(\frac{10^{7.42}\,{\rm K}}{10^7\,{\rm K}}\right)^{-1/3}\right] \\ &= 6.3\cdot 10^3\,{\rm erg\,g^{-1}\,s^{-1}}\,. \end{aligned} \tag{30}$$

b. Since the stars are on the ZAMS they are in equilibrium and I can say that the entropy no longer changes with time. Therefore, formula (11.19) in the book can be used to say that

$$\frac{\partial L_{\rm crit}}{\partial M_r} = \epsilon_{\rm crit}\,, \tag{31}$$

where $L_{\rm crit}$ is given by formula (11.29) in the book

$$L_{\rm crit} = \frac{64\pi G M_r \sigma_B T^3}{3\kappa}\left(\frac{\partial T}{\partial P}\right)_s\,. \tag{32}$$

Assuming that the gas is an ideal gas, for the central part

$$\left(\frac{\partial T}{\partial P}\right)_s = \frac{2}{5}\frac{T_c}{P_c} = \frac{2}{5}\frac{m_H\mu}{k_B\rho_c}\,. \tag{33}$$

Thus

$$L_{\rm crit} = \frac{128\pi m_H \mu G M_r \sigma_B T_c^3}{15\kappa k_B \rho_c}\,. \tag{34}$$

Now taking the derivative with respect to M_r I find that the critical energy generation rate is

$$\epsilon_{\rm crit} = \frac{128\pi m_H \mu G \sigma_B T_c^3}{15\kappa k_B \rho_c}\,. \tag{35}$$

c. To get a numerical expression for $\epsilon_{\rm crit}$ for the two masses, the appropriate expression for the opacity has to be used. Using the Kramers' Law expression the opacity would be

$$\kappa = 2.0\,{\rm cm^2\,g^{-1}}\left(\frac{\rho}{100\,{\rm g\,cm^{-3}}}\right)\left(\frac{T}{10^7\,{\rm K}}\right)^{-7/2}\,. \tag{36}$$

For the temperatures and densities of the 2 and 5 solar mass star the opacity would be, respectively,

$$\begin{aligned}\kappa &= 2.0\,{\rm cm^2\,g^{-1}}\left(\frac{10^{1.8}}{100\,{\rm g\,cm^{-3}}}\right)\left(\frac{10^{7.30}}{10^7\,{\rm K}}\right)^{-7/2} \\ &= 0.01\,{\rm cm^2\,g^{-1}} \end{aligned} \tag{37}$$

$$\begin{aligned}\kappa &= 2.0\,{\rm cm^2\,g^{-1}}\left(\frac{10^{1.2}}{100\,{\rm g\,cm^{-3}}}\right)\left(\frac{10^{7.42}}{10^7\,{\rm K}}\right)^{-7/2} \\ &= 0.11\,{\rm cm^2\,g^{-1}}\,. \end{aligned} \tag{38}$$

However, with such high temperatures and densities, the electron scattering becomes the dominant contribution to the opacity. Its value is independent of density and temperature and depends only on the hydrogen fraction of the star

$$\kappa_e = 0.2(1+X)\,\mathrm{cm^2\,g^{-1}}\,, \tag{39}$$

which is 0.34 $\mathrm{cm^2\ g^{-1}}$ for a star of solar composition. The electron scattering opacity is bigger than Kramers' opacity for both masses, thus this will be the appropriate opacity to use in calculating ϵ_{crit}.
With κ equal to 0.34 in eq. 35 the critical energy generation rate for the $2M_\odot$ star becomes

$$\epsilon_{\mathrm{crit}} = 2.8 \cdot 10^2\,\mathrm{erg\,g^{-1}\,s^{-1}}\,, \tag{40}$$

which is higher than the ϵ_{CNO} from eq. 29, thus the $2M_\odot$ star does not have a convective core.
For the $5M_\odot$ star the critical energy generation rate becomes

$$\epsilon_{\mathrm{crit}} = 2.5 \cdot 10^3\,\mathrm{erg\,g^{-1}\,s^{-1}}\,. \tag{41}$$

This value is lower than the ϵ_{CNO} from eq. 30, which implies that the $5M_\odot$ star does have a convective core.