HOW TO FOURIER FILTER YOUR REFERENCE SPECTRUM

We haven't covered this in class, but it a fact[1] that when you calculate a power spectrum using Fourier methods, as we do for our 21-cm line detections, you can do it two ways:

1. Measure time samples $E(t_j)$ and take the Fourier transform. For each frequency channel, the power spectrum is the square of the length of the complex vector. This is what we do.
2. Measure the autocorrelation function, which is $A(\tau) = \int E(t)E(t+\tau)\, dt$, and take the cosine Fourier transform of $A(\tau)$. (Here, τ represents time delay).

Here, in the second way, the important point is that—no matter which method you use—the power spectrum $P(f)$ is the Fourier transform of a quantity [which happens to be $A(\tau)$]. We emphasize: $P(f)$ is the cosine Fourier transform, *not* the square of the vector length as in (1). This means that we can recover $A(\tau)$ by taking the inverse cosine transform of $P(f)$.

Now, our reference spectra are generally smooth with no fine structure in frequency. The fine structure in frequency comes from the long time delays in $A(\tau)$. This means that the best way to smooth our measured $P(f)$ is to

1. Take the inverse transform of $P(f)$ to obtain $A(\tau)$.
2. Zero out $A(\tau)$ for $\tau \geq$ a suitable limit, chosen to ensure that no *real* structure in $P(f)$, which is all broad in frequency, is lost; this gets rid of the rapidly-varying noise.
3. Using this modified version of $A(\tau)$, take its Fourier transform to retrieve a smoothed, much less noisy version of $P(f)$.

So here's how to do it:

1. Derive $A(\tau)$

The power spectrum $P(f)$ is the cosine Fourier transform of $A(\tau)$, so $A(\tau)$ is the inverse cosine transform of $P(f)$. We could do this using the full-fledged expression for the Fourier transform, but it's much faster to use FFT. To force the FFT to take a cosine transform, we construct a symmetrized version of $P(f)$. In our digital spectrum we can write P_n instead of $P(f)$; while f runs from $-f_{Nyq}$ to $+f_{Nyq}$, n runs from 0 to $N-1$, where N is even and is equal to the number of spectral points in P_n.

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[1]This is a result of the *correlation theorem* for Fourier transforms.

Let's call the symmetrized version $P_{sym,j}$. It contains $J = 2N$ points. Consider j to run from $-N$ to $N-1$. Then we have

$$P_{sym,j} = P_n \ , \ for\ j = n\ and\ n = 0 \rightarrow N-1 \tag{1a}$$

$$P_{sym,j} = P_n \ , \ for\ j = -n\ and\ n = 1 \rightarrow N-1 \tag{1b}$$

These equations define $P_{sym,j}$ for the range $j = -(N-1) \ \rightarrow +(N-1)$, a total of $2N-1$ points; P_{sym} has an odd number of points, with perfect reflection symmetry about the origin.

We need to have an even number of points for the FFT, and we need to invent the point for $j = -N$. We should not set this invented point equal to zero; rather, we should set it equal to the nearest available point on the original P_n. Thus, set $P_{sym,-N} = P_{sym,-(N-1)}$ (which is the same as setting $P_{sym,-N} = P_{sym,+(N-1)}$).

2. CALCULATE $A(\tau)$

In IDL:

```
Psym_shift= shift( psym, N)
A = fft(Psym_shift, /inverse)
```

A has $2N$ elements. $\tau = 0$ is the first element and τ_{max} is the N^{th} element.

3. ZERO OUT $A(\tau)$ FOR $\tau \geq \tau_{max}$

Choose a τ_{max}. The smaller you make it, the less noisy will be the filtered spectrum. Set $A_{tau} = 0$ for $\tau \geq tau_{max}$. *Also, make sure* hata you keep $A(\tau)$ symmetric by zeroing $\pm\tau$ symmetrically. Call this modified version A_{mod}.

4. TRANSFORM BACK TO GET THE FOURIER-FILTERED $P(f)$

```
Pmod = fft( Amod)
```

At this point, the first N channels of `Pmod` are the filtered version of the original P. P_{mod} will be complex; the imaginary values should be zero to within machine accuracy. What you want is the real values.

5. BEFORE BARGING AHEAD, TRY THESE REALITY CHECKS!!!

1. Carry through the steps above but *don't* zero any elements of $A(\tau)$. You should regain the original spectrum!

2. After you derive P_{mod}, make sure the imaginary values are zero. If they are not, you're not zeroing $A(\tau)$ symmetrically for $\pm\tau$.