SPECIFIC INTENSITY: THE FOUNT OF ALL KNOWLEDGE!
or...
I, T_B, T_{SYS}, T_{RCVR}, S, and $\tau\Delta\nu$

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1. INTRODUCTION

The cosmic radiation that we measure differs from radiation produced by humankind. Humankind's transmitters usually produce radiation at specific frequencies; this radiation is modulated with an information-containing signal of some sort, such as rap music (here we use the term "information" loosely). In contrast, cosmic radiation is not produced by slow modulation of a carrier frequency with an information-containing signal generated by an intelligent being. Rather, cosmic radiation is produced by random processes, of which there are three common types: collisions between electrons and protons in a gas at temperature T; spiraling of relativistic electrons in a magnetic field; and transitions between energy levels of an atom or molecule. The former two generate radiation over a wide spectrum of frequencies while the latter generates a narrow spectrum called a *spectral line*. The radiation from the former two is called *continuum radiation* because its frequency spectrum is very broad and featureless. For these random processes, we can describe the spectrum as a sum of Fourier components; all frequency components in a small bandwidth have comparable power and random electrical phases, and the total power varies with frequency to give the spectrum we observe.

Continuum radiation is produced by all astronomical objects. For example, stars produce continuum radiation that is strongest at optical or IR frequencies. Galaxies and clouds of hot gas also produce optical and IR continuum radiation. They also produce continuum radiation at radio frequencies, and this is usually much less intense than the optical continuum radiation. Some sources have relativistic electrons, and these sometimes produce powerful radiation at radio, X-ray, and gamma-ray frequencies.

2. SPECIFIC INTENSITY

In the above paragraph we compared radio and optical continuum radiation by their intensities, and indeed the intensity is one of the two basic properties of continuum radiation. The other is the direction from which it comes. But this concept of direction needs a bit of elaboration: all sources occupy a finite solid angle, and thus a *range* of directions. *There is no such thing as a true point source!*

For example, the Sun is 0.5° in diameter and occupies a solid angle of 0.79 deg^2. The Sun moves across the sky, so it has a direction. We need to combine these two properties—intensity, size, and direction—to obtain a general quantity with which we can characterize a source of light.

This general quantity is called the *specific intensity*, always denoted by the symbol $I(\nu)$. This has units of (are you ready?)

$$units\ of\ I(\nu) = J\ s^{-1}\ m^{-2}\ Hz^{-1}\ ster^{-1} \tag{1a}$$

or

$$units\ of\ I(\nu) = Watts\ m^{-2}\ Hz^{-1}\ ster^{-1} \tag{1b}$$

or, in my favorite units

$$units\ of\ I(\nu) = ergs\ s^{-1}\ cm^{-2}\ Hz^{-1}\ ster^{-1} \tag{1c}$$

In these equations, the area (m^2 or cm^2) is the area of the receiving surface, which in our case is a telescope. Moreover, it is also clear that $I(\nu)$ is a function of *direction*; we've not written this above to keep things simpler, but for a complete specification we need to include this directional dependence. In astronomy, we usually specify the direction in terms of right ascension and declination (ra, dec) or (α, δ), so we need to write

$$I(\nu, \alpha, \delta) \tag{1d}$$

for the complete specification.

2.1. Specific Intensity: the same as angular surface brightness

The fundamental point: specific intensity tells the power coming from each little solid angle of the source. Think of it as *angular surface brightness.* When we make an image (or map) of a source, such as the Sun or a galaxy, it's like a photograph. This image is an image of specific intensity—or angular surface brightness—at a particular frequency (or, more properly, over a range of frequencies centered at a particular frequency). An image tells you nothing about the power per unit *linear* surface area; rather, it tells you about the power per *angular* surface area.

When discussing angular surface brightness it's most familiar to think of an optical image. When we make that image, for example in the optical by taking a CCD image, we collect energy from the source to move those electrons around. CCD's have lots of little pixels. The amount of energy collected by each pixel is

$$dE(\nu) = I(\nu)\ dA\ d\nu\ dt\ d\Omega, \tag{2}$$

Here, $I(\nu)$ is the specific intensity, dA is the area of the pixel, $d\nu$ is the bandwidth (which is usually determined by a filter, say a red filter, in front of the CCD), and $d\Omega$ is the solid angle on the sky seen by that pixel. Of these quantities, $d\Omega$ is the hardest to understand because it depends on the focal ratio of the telescope; a "fast" telescope with a low focal ratio has each pixel seeing a large solid angle, so the pixel collects lots of energy compared to the case of a large focal ratio. If you're into photography, you're familiar with all this.

3. A FUNDAMENTAL THEOREM: SPECIFIC INTENSITY IS CONSTANT ALONG A RAY PATH(!)

An important, and surprising, and counterintuitive theorem: the specific intensity is constant along a ray path! One can prove this with elegant mathematical formalism, but let's consider two simple cases. First, however, note that we talk about a "ray path". Talking about ray paths means, implicitly, that wavelengths are short so that diffraction plays no role. Thus this theorem hold only when angular sizes are much larger than $\frac{\lambda}{D}$.

3.1. Specific intensity is independent of distance!

Remarkably, $I(\nu)$ is *independent of distance of an object!* Consider the Sun as an example. Each square meter of the Sun's surface emits some amount of power. As we move further away in distance D, that power becomes diluted by the geometrical factor $\frac{1}{D^2}$. However, the number of square meters in a steradian increases as D^2. Because $I(\nu)$ specifies the power *per steradian*, the distance D cancels out!

3.2. Specific intensity at the telescope focal plane is the same as at the source!

Suppose you observe the Sun with a telescope. The specific intensity at the focal plane of the telescope, I_{fp}, is identical to that at the Solar surface $I_\odot$!

Look at Figure 1, which exhibits a simple optical telescope with a lens. The lens has focal length d, so the focal plane (fp) is separated from the lens by distance d, and the lens occupies solid angle $\Omega_{lens} = \frac{\pi R_{lens}^2}{d^2}$ as seen from the fp. Similarly, the Sun lies distance D from the lens and occupies solid angle $\Omega_\odot = \frac{\pi R_\odot^2}{D^2}$ as from the lens. The area of the Sun on the focal plane is $A_\odot = \pi R_\odot^2 \frac{d^2}{D^2}$.

As we determined above, the specific intensity at the lens I_{lens} is equal to that at the Sun's

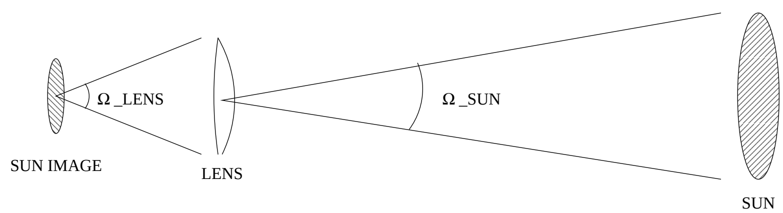

Fig. 1.— Observing the Sun with a telescope, illustrating that the specific intensity I is unchanged along a ray path.

surface $I_\odot$. Thus the total power picked up by the lens is this specific intensity times the lens area times the solid angle of the Sun, i.e.

$$P = I_\odot(\pi R_{lens}^2)\Omega_\odot \tag{3}$$

All of this power goes to the image on the focal plane. The specific intensity at the fp is this power P divided by the product (the area of the image times the solid angle of the lens), i.e.

$$I_{fp} = \frac{P}{\left(\pi R_\odot^2 \frac{d^2}{D^2}\right)\left(\frac{\pi R_{lens}^2}{d^2}\right)} \tag{4}$$

Plug equation 3 into this and all factors cancel, leaving

$$I_{fp} = I_{\odot} \quad (!) \tag{5}$$

Again, we reiterate that there is one assumption here, namely that diffraction plays no role—i.e. that the telescope easily resolves the source in angle. For our radio telescope observing the Sun, this would require the telescope beam size $\ll$ the angular size of the Sun.

4. BLACKBODY RADIATION AND BRIGHTNESS TEMPERATURE

A blackbody at temperature T emits specific intensity $I(\nu)$ which is given by the usual formula for blackbody emission, known as Planck's law:

$$I(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}. \tag{6}$$

In radio astronomy we always have $\frac{h\nu}{kT} \ll 1$, which means the Rayleigh-Jeans approximation applies and the complicated Planck law reduces to

$$I(\nu) = \frac{2kT}{\lambda^2} \tag{7}$$

4.1. Brightness temperature

The units of $I(\nu)$ are long and complicated: Watts m^{-2} Hz^{-1} ster^{-1}. Radio astronomers—being as lazy as anyone else—notice the direct proportionality between $I(\nu)$ and T and prefer to use T in place of $I(\nu)$, with its simpler units of Kelvins. Thus, instead of angular surface brightness or specific intensity, radio astronomers speak of the *brightness temperature* of an object and denote it by the symbol T_B. Thus we have...

$$T_B = \frac{I(\nu)\lambda^2}{2k} \tag{8}$$

What's the brightness temperature of the Sun? At optical wavelengths it's just about equal to the surface temperature, i.e. $T_B \sim 6000$ K. At radio wavelengths it happens to be much higher because radio telescopes see the Sun's coronal gas, which is much hotter—in the millions of Kelvins—and also because they see synchrotron radiation from sunspots and flares.

More generally, astronomical sources aren't always black bodies. Thus the brightness temperature of a source is not necessarily equal to its *physical* temperature. Nevertheless, we always retain the definition embodied in equation 8. The source emits some specific intensity $I(\nu)$, and we define T_B as that temperature that reproduces $I(\nu)$. If the source is indeed a thermally-emitting blackbody, then T_B is independent of frequency; if not, then $T_B = T_B(\nu)$.

5. BRIGHTNESS TEMPERATURE AND ANTENNA TEMPERATURE.

In the discussion relevant to equation 2, we referred to a CCD and its pixels. Let us rewrite equation 2 using the Rayleigh-Jeans approximation—which makes it specific to radio astronomy. Each pixel collects a power per Hz $P(\nu)$ equal to

$$P(\nu) = \frac{2kT_B(\nu)}{\lambda^2}\ dA\ d\Omega \tag{9}$$

Now consider what a "pixel" is for a diffraction-limited system as we have in radio astronomy. The pixel is the "feed", which has area dA and responds to solid angle $d\Omega$; in a well-engineered system, the primary mirror occupies the same solid angle $d\Omega$. With any diffraction-limited telescope, including the feed (which can be considered as just another telescope), these are related: the diffraction angle $\theta \propto \frac{\lambda}{size}$, so the solid angle $\Omega \propto \frac{\lambda^2}{area}$. It so happens that the proportionality constant is unity, $\Omega = \frac{\lambda^2}{area}$ (see §6 for more elaboration). Thus above, in equation 9, the product $(dA\ d\Omega)$ is equal to λ^2. The factor λ^2 appears in both the numerator and denominator, so equation 9 reduces to

$$P(\nu) = 2kT_B(\nu) \tag{10}$$

Now we do the same trick that we did in §4.1 for brightness temperature. Namely, we realize that radio astronomers are lazy and that the units of $P(\nu)$ are cumbersome [but not so much as $I(\nu)$!]. Then we simply write

$$2kT_A(\nu) = P(\nu) \tag{11}$$

which allows us to write the simple, and *physically meaningful* equation

$$T_A(\nu) = T_B(\nu) \tag{12}$$

This equation is *physically meaningful* because the antenna temperature—the power per Hz picked up from the source—is equal to the brightness temperature of the source! With one caveat: we have implicitly assumed that the source solid angle, when projected onto the focal plane, is larger than the area of the feed. This is equivalent to assuming that the telescope resolves the source, i.e. that the angular size of the source exceeds the telescope beamwidth on the sky.

5.1. Antenna temperature for an unresolved source

A telescope (in radio astronomical parlance, an "antenna") picks up a certain fraction of brightness of the source, producing the *antenna temperature* T_A. If a blackbody source at temperature T_B fills the solid angle of the antenna beam, then section 5 applies and $T_A = T_B$.

If the source is *smaller* than the antenna beam, then some of the beam sees the cold sky that lies outside the boundaries of the source. In this case we clearly have $T_A < T_B$. To be a bit more quantitative, if the source has solid angle Ω_s and the antenna beam Ω_b, then the antenna temperature is

$$T_A \approx T_B \frac{\Omega_s}{\Omega_b + \Omega_s} \tag{13}$$

In other words, if the source is tiny, $T_A \ll T_B$; as the source gets larger, T_A increases to the upper limit T_B. These solid angles are a bit ill-defined, which accounts for the $\approx$.

With this equation, we see that if a source is "fully resolved" with $\Omega_b \ll \Omega_s$, then the antenna temperature is equal to the brightness temperature and we can make a high-fidelity map of the radio angular surface brightness. If the source is not fully resolved, the angular surface brightness variations within our map are reduced by the convolving effect of the beam with the source.

6. FLUX DENSITY

The *total* output of the Sun is large. That is, its apparent luminosity or *flux density* is large. The flux density $S(\nu)$ is equal to the integral of the angular surface brightness over the solid angle is large. The total power density from the whole source incident on the telescope is just

$$S(\nu) = \int I(\nu) d\Omega = \frac{2k \int T_B d\Omega}{\lambda^2} \ , \tag{14}$$

The units of $S(\nu)$ are Watts m^{-2} Hz^{-1}.

Again, to avoid these complicated units, radio astronomers replace this with the *Jansky*, which is equal to 10^{-26} Watts m^{-2} Hz^{-1}. The strongest radio source in the sky, other than the Sun, is the $\sim$ 300-yr old supernova remnant Cas A, with $S \sim 2000$ Jy. There are a few hundred sources with $S \gtrsim 1$ Jy, and uncounted numbers at weaker fluxes. With modern telescopes, it is almost trivial to measure fluxes at the 1 mJy level.

6.1. Flux Density and antenna temperature.

For a source whose brightness temperature T_B is constant over a finite solid angle Ω_s, we can remove the integral sign from equation (14) and write...

$$S(\nu) = I(\nu)\Omega_s = \frac{2kT_B\Omega_s}{\lambda^2} \ , \tag{15}$$

or

$$T_B\Omega_s = \frac{\lambda^2}{2k}S(\nu) \tag{16}$$

Further, if $\Omega_s \ll \Omega_b$ then we can plug this into equation (13) and write...

$$kT_A = \left[\frac{\lambda^2}{\Omega_b}\right]\frac{S(\nu)}{2} \tag{17}$$

Now, this is an interesting equation. *Look at the factor in square brackets: the numerator and denominator are proportional to each other—with the proportionality constant equal to the telescope area!* The telescope beam solid angle $\Omega_b \propto$ beamwidth$^2 \propto (\frac{\lambda}{2R})^2$, where R is the telescope radius. In fact, as it turns out, it happens to be an exact, fundamental result of electromagnetic theory that

$$\Omega_b = \frac{\lambda^2}{\pi R^2} \tag{18}$$

This is for a circular aperture; for an aperture of arbitrary shape and area A, it is more generally true that...

$$\Omega_b = \frac{\lambda^2}{A} \tag{19}$$

Plugging this into equation 17, we get

$$kT_A = \frac{S(\nu)A}{2} \tag{20}$$

In words: The total power per Hz intercepted by the antenna is $\frac{S(\nu)A}{2}$ (think of the radio photons like raindrops falling on the telescope). Also, kT is the power per Hz available from a resistor at temperature T (as we discuss in §7). So the antenna temperature is just what you'd expect—it's equal to the power per Hz collected by the telescope.

Wait a minute! What about the factor of $\frac{1}{2}$? That's because of polarization. There are two orthogonal polarizations, each containing half the power. Our equations refer to only a single polarization, which picks up only half the total power[1]. If we had a dual-polarized system then each channel would have the above antenna temperature, and the combination gives the full power per Hz from the source.

7. RELATIONSHIP BETWEEN THE POWER FROM A RESISTIVE LOAD AND A BLACKBODY.

We can derive this relationship in two different ways. The complicated way involves using statistical mechanics. The simple way involves a thermodynamical argument, in which the complications of statistical mechanics have been incorporated into the concepts of thermodynamical equilibrium. We, of course, use the simple way.

Imagine an antenna immersed in a blackbody cavity of temperature T_B, as in figure 2. We saw above in §5 that $T_A = T_B$, and specifically from equation 10 and ensuing discussion that each polarization picks up power $P = kT_B$ erg sec^{-1} Hz^{-1}.

Using a coax cable, connect one polarization channel to a matched resistive load that resides in a second blackbody cavity at the same temperature T_B, which happens to be equal to T_B; obviously, the load is also at temperature T_B. The load, being perfectly matched, absorbs all the power collected by the antenna.

Thus power P will be transferred to this second cavity from the first cavity. However, thermodynamics says that this is impossible, because the two cavities have identical temperatures. Therefore the resistor must also send the same amount of power to the antenna. Thus, the power available from the resistor is also $P = kT\Delta\nu$ erg sec^{-1} Hz^{-1}.

Summary: A load at temperature T generates the same amount of power in an electrical circuit that a well-matched antenna absorbs from a blackbody cavity at temperature T. That is, it generates $P = kT\Delta\nu$ Watts Hz^{-1} (or erg s^{-1} Hz^{-1}).

8. RECEIVER TEMPERATURE, SYSTEM TEMPERATURE

The power received by a radio telescope is exceedingly small, and to detect it we must amplify by about 100 db. With this large amplification, internal noise in the system is also amplified. This internal noise is called *receiver noise* and is equivalent in all respects to the radiation coming in from the sky: it is continuum in nature and generated by random processes. It is convenient to

[1]This applies if the source is unpolarized. If it is polarized, then the power doesn't split equally between the two polarizations.

Fig. 2.— A perfectly matched telescope and load, each in its own blackbody cavity; the cavities are at the same temperature.

measure it with the same units as the celestial radiation, namely temperature. So we specify the receiver noise as the *receiver temperature*, denoted by the symbol T_R. Just as for any thermal noise source, the noise power is $P = kT\Delta\nu$ (see § 7).

In a well-designed system, all of the receiver noise is generated in the *first* amplifier. The receiver temperature T_R is not the *physical* temperature of the amplifier. Modern amplifiers have receiver temperatures that are much smaller than the actual physical temperatures. However, their receiver temperatures go even lower when they are cooled. Most amplifiers in use for radio astronomy are cooled, usually to temperature $\sim$ 15 K (commercial refrigerators are cheap and reliable) and sometimes to temperature $\lesssim$ 4 K ("Helium" temperatures—not cheap, much less reliable).

The telescope collects the incoming power from the sky, specified by the antenna temperature T_A. This combines linearly with the power generated by the receiver and their sum is called the total *system temperature* $T_S = T_R + T_A$. Usually, $T_R \gg T_A$ and detecting the weak "signal" T_A in the presence of the large receiver "noise" T_R is a challenge. (In fact, both have the same character: randomly-varying voltage *vs.* time, which is in fact the character of what we commonly call noise).

9. SENSITIVITY, INTEGRATION TIME, AND BANDWIDTH

The uncertainty in the measurement of system temperature is about $\Delta T_S = \frac{T_S}{\sqrt{\tau\Delta\nu}}$, where τ is the integration time in seconds. The easy way to understand this is to realize that with bandwidth $\Delta\nu$ you attain one independent sample of the random fluctuating power for each independent time interval. The signal is statistically independent over time interval $\frac{1}{\Delta\nu}$. This means that the number of independent samples is $N = \tau\Delta\nu$. The quantity $\tau\Delta\nu$ is known as the *time-bandwidth product* and, in essence, is equal to the number of independent samples of the signal. When sampling a random function the r.m.s. uncertainty is always equal to the function itself divided by $\sqrt{N}$ (this is known as "root-N statistics").

To reiterate: the *fractional* uncertainty in the measurement of system temperature is

$$\frac{\Delta T_S}{T_S} = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{\tau\Delta\nu}} \tag{21}$$

Thus, the longer we integrate, the more sensitive our measurement. The wider the bandwidth, the more sensitive. For continuum work, the bandwidth is limited only by our instrumentation; for spectral line work, the bandwidth is limited by the width of the spectral line.

And, *most importantly*: to get a small error in the quantity of interest—which is the antenna temperature T_A—we need to get T_S as small as possible. This, in turn, means getting the receiver noise temperature T_R as small as possible. *The receiver temperature is the most important parameter for determining the sensitivity of a radio astronomical receiver.* Or, for that matter, any other receiver, whether it be radio or TV. You need the best signal to noise ratio you can get!

10. SOME PRACTICAL INFORMATION ON SIGNAL LEVELS

Suppose you are observing a bandwidth $\Delta\nu = 1$ MHz and your system temperature is $T_S = 100$ K. This gives r.f. power $P = kT_S\Delta\nu$; with $k = 1.4 \times 10^{-23}$ Watt-sec K^{-1} this gives $P = 1.4 \times 10^{-15}$ Watts. This is quite small!

Most lab equipment, such as digital sampleres, needs $P \gtrsim 1$ milliWatt, or $P \gtrsim 10^{-3}$ Watts. So we need to amplify the signal by a factor of 10^{12} in power—that's 120 dB.

While the Watt is a suitable power measurement, we often want to express power in a logarithmic unit. The standard one is the dBm: the power expressed in dB relative to 1 milliWatt. Thus, our r.f. power is about -118.5 dbM and we need 118.5 dB amplification to give 0 dBm. Using the relation $P = \frac{V^2}{R}$, for our 50 Ohm cable a power of 0 dBm corresponds to about .2 Volts.