MUELLER MATRIX PARAMETERS FOR RADIO TELESCOPES AND THEIR OBSERVATIONAL DETERMINATION

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ABSTRACT

Modern digital crosscorrelators permit the simultaneous measurement of all four Stokes parameters. However, the results must be calibrated to correct for the polarization transfer function of the receiving system. The transfer function for any device can be expressed by its Mueller matrix. We express the matrix elements in terms of fundamental system parameters that describe the voltage transfer functions (known as the Jones matrix) of the various system devices in physical terms and thus provide a means for comparing with engineering calculations and investigating the effects of design changes. We describe how to determine these parameters with astronomical observations. We illustrate the method by applying it to some of the receivers at the Arecibo Observatory.

Subject headings: polarization — instrumentation: polarimeters — techniques: polarimetric

1. INTRODUCTION

The polarization response of a single-dish radio telescope is usually described in terms of the native polarization of the feed. For example, a dual-linear polarized feed provides two outputs ("channels") which are orthogonal linear polarizations; the sum is Stokes I and the difference Stokes Q. Correlating the two outputs with zero phase difference provides the other linearly polarized Stokes parameter U, and correlating with 90° phase difference provides the circularly polarized Stokes parameter V. Similarly, with a dual-circular feed the sum and difference provide I and V, and the correlated outputs provide Q and U. However, these statements are only approximate. Feeds are almost never perfect, so their polarizations are only approximately linear or circular; moreover, some feeds, such as turnstile junctions, can have two orthogonal polarizations with arbitrary ellipticity that is frequency-dependent. And generally speaking, no feed has two outputs that are perfectly orthogonal.

The feed's response is modified by the electronics system, which introduces its own relative gain and phase differences between the two channels. These must be calibrated relatively frequently because they can change with time. For example, if the feed is perfect and the relative gain differs, then the difference between the two channels is nonzero for an unpolarized source, making the source appear to be polarized. For a perfect linearly polarized feed, if the relative phase of the two channels differs, then a linearly polarized source appears to be circularly polarized. Thus two quantities, the relative gain and phase, must be calibrated. This is most effectively done by injection of a correlated noise source into the two feed outputs or, alternatively, by radiating a noise source into the feed with a polarization that provides equal amplitudes in the feed outputs.

The modification of the four Stokes parameters by these system components is most generally described by a 4×4 matrix. This matrix is known as the Mueller matrix (Tinbergen 1996). Along with every Mueller matrix goes a Jones matrix, which describes the transfer function for the voltages. We express the Jones matrix elements in terms of complex voltage coupling coefficients. We consider three such matrices: one describes the polarization state of the feed (circular, elliptical, or linear); one the nonorthogonality of the polarized outputs; and one the relative gain and phase of the electronics system. We calculate the associated Mueller matrices and describe how to solve for the coupling coefficients from astronomical measurements. We illustrate the method by applying it to some of the systems at Arecibo Observatory[1].

We begin by reviewing the basic theoretical concepts and developing the structure of our treatment. §2 introduces the Stokes parameters, Mueller matrix, and Jones vector and matrix. §3 defines the Mueller matrices for the different effects we describe, including the mechanical rotation of the feed with respect to the sky, the imperfections in the feed, the feed, and the amplifier chain. §4 discusses the combined effects of the receiver components and calculates that matrix product. §5 describes the technique for evaluating the matrix elements from observations. §6 describes how to apply the derived Mueller matrix to correct for the instrumental effects. Finally, §7 provides illustrative results for the two 21-cm line receivers at Arecibo Observatory. This paper is a more general and succinct version of Arecibo Technical and Operations Memo 2000-04, which covers all of Arecibo's receivers.

2. STOKES, MUELLER, AND JONES

The basic reference for our discussion of the fundamentals is the excellent book on astronomical polarization by Tinbergen (1996). A more mathematical and fundamental reference is Hamaker, Bregman, and Sault (1996), which the theoretically-inclined reader will find of interest. In the following we make several unproven statements and assertions about Mueller and Jones matrices;

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[1]The Arecibo Observatory is part of the National Astronomy and Ionosphere Center, which is operated by Cornell University under a cooperative agreement with the National Science Foundation

the explanations and justifications can be found in the abovementioned references.

2.1. The Stokes and Jones vectors

The fundamental quantities are the four Stokes parameters (I, Q, U, V), which we write as the 4-element Stokes vector

$$\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} . \tag{1}$$

The Stokes parameters are time averages of electric field products; we use the terms "voltage" and "electric field" interchangeably because the radio telescope's feed converts one to the other. The Jones vector represents the fields as orthogonal linear polarizations (E_X, E_Y), and considers them as complex to account for their relative phase:

$$\mathbf{J} = \begin{bmatrix} E_X \\ E_Y \end{bmatrix} . \tag{2}$$

Instructive special cases include pure linear and pure circular polarization. Orthogonal linear polarizations are, obviously, $(\mathbf{J_X}, \mathbf{J_Y}) = ([1,0],[0,1])$ (In the text, we write these vectors as rows instead of columns for typographical purposes). Orthogonal circular polarizations are $(\mathbf{J_L}, \mathbf{J_R}) = \left(\frac{[1,i]}{\sqrt{2}}, \frac{[i,1]}{\sqrt{2}}\right)$, where $i = \sqrt{-1}$. Orthogonal polarizations satisfy, for example, $J_X \overline{J_Y} = 0$, where the bar over a symbol indicates the complex conjugate and all products are time averages.

It is straightforward to relate the Stokes parameters to the components of the Jones vector:

$$I = E_X \overline{E_X} + E_Y \overline{E_Y} \tag{3a}$$

$$Q = E_X \overline{E_X} - E_Y \overline{E_Y} \tag{3b}$$

$$U = E_X \overline{E_Y} + \overline{E_X} E_Y \tag{3c}$$

$$iV = E_X \overline{E_Y} - \overline{E_X} E_Y . \tag{3d}$$

2.2. The Mueller and Jones matrices

When the fields pass through some device, such as a feed or an amplifier, they suffer amplitude and phase changes. These modify the Stokes parameters. The Mueller matrix is the transfer function between the input and output of the device:

$$\mathbf{S_{out}} = \mathbf{M} \cdot \mathbf{S_{in}} \ . \tag{4}$$

The Mueller matrix is, in general, a 4×4 matrix in which all elements may be nonzero (but they are not all independent). In the usual way, we write

$$\mathbf{M} = \begin{bmatrix} m_{II} & m_{IQ} & m_{IU} & m_{IV} \\ m_{QI} & m_{QQ} & m_{QU} & m_{QV} \\ m_{UI} & m_{UQ} & m_{UU} & m_{UV} \\ m_{VI} & m_{VQ} & m_{VU} & m_{VV} \end{bmatrix} \ . \tag{5}$$

The matrix elements are just the partial derivatives, for example

$$m_{VQ} = \left. \frac{\partial V_{out}}{\partial Q_{in}} \right|_{I_{in}, U_{in}} \ . \tag{6}$$

Every Mueller matrix has its Jones matrix counterpart; the Jones matrix is the transfer function for the voltages. We defer further discussion of Jones matrices until our treatment of three specific cases of interest for radio astronomical systems.

3. THE RADIOASTRONOMICAL RECEIVER COMPONENTS AND THEIR MATRICES

In this section, we consider the Mueller matrices of devices that are encountered by the incoming radiation on its way from the sky to the correlator output. We consider three devices. The first device encountered by the incoming radiation is the telescope ($\mathbf{M_{sky}}$), which mechanically rotates the feed with respect to the sky. The second device is the feed, which we split into two parts. The first part ($\mathbf{M_F}$) has the ability to change the incoming linear to any degree of elliptical polarization; by design, feeds are intended to produce either pure linear or pure circular, but in practice the polarization is mixed, i.e. elliptical. The second part ($\mathbf{M_{IFr}}$) describes imperfections in such a feed, specifically the production of nonorthogonal polarizations. The third device is the amplifier chain ($\mathbf{M_A}$).

We describe the Jones matrix of the incoming radiation in linear polarization, as in equation 2. However, the feed matrix $\mathbf{M_F}$ can radically change the polarization state; for example, as we shall see a dual-circular feed changes the order of the Stokes parameters in equation 1. Thus the signal voltages, after going through $\mathbf{M_F}$, are not intuitively described as (E_X, E_Y), because the (X, Y) connote linear polarization. Therefore, for the voltages after the output of $\mathbf{M_F}$, we will use the symbols (E_A, E_B), or simply (A, B), to emphasize the fact that the state of polarization can be arbitrary.

We assume that the remaining device matrices $\mathbf{M_{IFr}}$ and $\mathbf{M_A}$ produce closely-matched replicas of the input Stokes parameters because, by design (hopefully!), the imperfections in the feed are small. We will retain only first-order products of these imperfections.

3.1. Mueller matrix relating the radio source to the receiver input

Astronomical continuum sources tend to have linear polarization but very little circular polarization; we assume the latter to be zero. Moreover, we express the source polarization as a fraction of Stokes I. Thus,

$$\mathbf{S_{src}} = \begin{bmatrix} 1 \\ Q_{src} \\ U_{src} \\ 0 \end{bmatrix} . \tag{7}$$

A linearly polarized astronomical source has Stokes (Q_{src}, U_{src}) defined with respect to the north celestial pole (NCP). The source polarization is conventionally specified in terms of fractional polarization and position angle with respect to the NCP, measured from north to east. We have

$$Q_{src} = P_{src} \cos 2PA_{src} \tag{8a}$$

$$U_{src} = P_{src} \sin 2PA_{src} \tag{8b}$$

$$P_{src} = (Q_{src}^2 + U_{src}^2)^{1/2} \tag{8c}$$

$$PA_{src} = 0.5 \tan^{-1} \left(\frac{U_{src}}{Q_{src}} \right) \tag{8d}$$

As we track a source with an alt-az telescope, the parallactic angle PA_{az} of the feed rotates on the sky. PA_{az} is defined to be zero at azimuth 0 and increase towards the east; for a source near zenith, $PA_{az} \sim az$, where az is the azimuth angle of the source. The Stokes parameters seen by the telescope are (Q_{sky}, U_{sky}), and are related to the source parameters by

$$\mathbf{M_{SKY}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2PA_{az} & \sin 2PA_{az} & 0 \\ 0 & -\sin 2PA_{az} & \cos 2PA_{az} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{9}$$

The central 2×2 submatrix is, of course, nothing but a rotation matrix.

For an equatorially-mounted telescope, PA_{az} doesn't change unless the feed is mechanically rotated with respect to the telescope. This was the case with the NRAO 140-foot telescope[2], which was one of the several systems we used during the genesis of this work. This relative rotation between feed and telescope can produce unintended changes in the feed response to astronomical sources, which makes such telescopes inherently less accurate for polarization measurement. The 140-foot telescope is the last of the great equatorial radio telescopes and with its recent closure a detailed discussion of these matters has become irrelevant; the remainder of the present paper considers alt-az telescopes exclusively.

3.2. Mueller matrix for a perfect feed providing arbitrary elliptical polarization

The feed modifies the incoming voltages with its Jones matrix. Suppose that the feed mixes incoming linear polarizations with arbitrary phase and amplitude, keeping the total power constant and retaining orthogonality; this makes it a perfect feed that responds to elliptical polarization. Following Stinebring (1982) and Conway and Kronberg (1969), we write the transfer equation for the feed as

$$\begin{bmatrix} E_{A,out} \\ E_{B,out} \end{bmatrix} = \begin{bmatrix} \cos\alpha & e^{i\chi} \sin\alpha \\ -e^{-i\chi} \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} E_{X,in} \\ E_{Y,in} \end{bmatrix} . \tag{10}$$

The feed can completely alter the polarization state, so the output Jones voltages are more intuitively described by subscripts (A, B) instead of (X, Y), which connote linear polarization. Here α is the amount of coupling into the orthogonal polarization and χ is the phase angle of that coupling. For example, for a native linear feed $\alpha = 0$ and $\chi = 0$; for a native circular feed $\alpha = 45°$ and $\chi = 90°$. Using this with equation 3, we find

[2]The National Radio Astronomy Observatory is a facility of the National Science Foundation operated under cooperative agreement by Associated Universities, Inc.

$$\mathbf{M_F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha \cos\chi & \sin 2\alpha \sin\chi \\ 0 & -\sin 2\alpha \cos\chi & \cos^2\alpha - \sin^2\alpha \cos 2\chi & -\sin^2\alpha \sin 2\chi \\ 0 & -\sin 2\alpha \sin\chi & -\sin^2\alpha \sin 2\chi & \cos^2\alpha + \sin^2\alpha \cos 2\chi \end{bmatrix} . \tag{11}$$

Notice that in the right-bottom 3×3 submatrix, the off-diagonal transposed elements are of opposite sign for two of the three pairs and the same sign for one. This is not an algebraic error!

Some instructive special cases include:

(1) A dual linear feed: $\alpha = 0$, $\chi = 0$, and M_F is diagonal.

(2) A dual linear feed rotated 45° with respect to (X, Y): $\alpha = 45°$, $\chi = 0$, and

$$\mathbf{M_F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{12}$$

As expected, this interchanges Stokes Q and U, together with a sign change as befits rotation.

(3) A dual linear feed rotated 90° with respect to (X, Y): $\alpha = 90°$, $\chi = 0$, and

$$\mathbf{M_F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{13}$$

As expected, this reverses the signs of Stokes Q and U.

(4) A dual circular feed: $\alpha = 45°$, $\chi = 90°$, and

$$\mathbf{M_F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} . \tag{14}$$

The combination $(\alpha = 45°, \chi = 90°)$ permutes the order of the Stokes parameters in the output vector, making it $(I, V, U, -Q)$. $(E_A\overline{E_A} - E_B\overline{E_B})$ provides Stokes V, instead of the Q written in equation 3; in other words, it makes the feed native dual circular. With respect to linear polarization, $\alpha = 45°$ has the same effect as in case **(2)**, namely to interchange (Q, U) and change the sign of Q, because it is equivalent to a feed rotation of 45°.

(5) A dual-elliptical feed: arbitrary α, $\chi = 90°$. If $\chi = 90°$ then orthogonal linear inputs produce orthogonal elliptical outputs with the ellipticity voltage ratio equal to $\tan\alpha$ (see Tinbergen 1996, Figure 2.1). Thus $\alpha = 0°$ passes the linear polarizations without modification, $\alpha = 90°$ reverses the sign of the position angle, $\alpha = 45°$ produces orthogonal circular polarizations, and other values of α produce orthogonal elliptical polarizations.

3.3. An important restriction: we set $\chi = 90°$

Generally speaking, we prefer either pure linear or pure circular feeds, i.e. we prefer either the combination $(\alpha, \chi) = (0°, 0°)$ or $(\alpha, \chi) = (45°, 90°)$ in $\mathbf{M_F}$. If a feed is designed to produce circular polarization with $(\alpha, \chi) = (45°, 90°)$ and instead produces elliptical polarization whose major axis is aligned with the X direction, then $\chi = 90°$ but $\alpha \neq 45°$. If the ellipse is not aligned with X, then $\chi \neq 90°$, but this is equivalent to having $\chi = 90°$ and physically rotating the feed. Thus, without loss of generality, we can take $\chi = 90°$ (also see §3.4). This leads to great simplification in $\mathbf{M_F}$, whose restricted form becomes

$$\mathbf{M_{Fr}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & 0 & \sin 2\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin 2\alpha & 0 & \cos 2\alpha \end{bmatrix} . \tag{15}$$

3.4. An imperfect feed

Again we follow Stinebring (1982) and Conway and Kronberg (1969), and represent the imperfections of a feed by the Jones matrix

$$\begin{bmatrix} E_{X,out} \\ E_{Y,out} \end{bmatrix} = \begin{bmatrix} 1 & \epsilon_1 e^{i\phi_1} \\ \epsilon_2 e^{-i\phi_2} & 1 \end{bmatrix} \begin{bmatrix} E_{X,in} \\ E_{Y,in} \end{bmatrix} . \tag{16}$$

Here the ϵ's represent undesirable cross coupling between the two polarizations; for example, this might be caused by the two linear probes not being exactly 90° apart. The ϕ's are the phase angles of these coupled voltages. This equation assumes that the feed is "good", meaning that we need retain only first-order terms in ϵ (which makes the diagonal elements unity); however, for the moment we allow the phases to be arbitrary.

After a little algebra, we find the matrix for the **I**mperfect **F**eed to be

$$\mathbf{M_{IF}} = \begin{bmatrix} 1 & 0 & \Sigma\epsilon\cos & \Sigma\epsilon\sin \\ 0 & 1 & \Delta\epsilon\cos & \Delta\epsilon\sin \\ \Sigma\epsilon\cos & -\Delta\epsilon\cos & 1 & 0 \\ \Sigma\epsilon\sin & -\Delta\epsilon\sin & 0 & 1 \end{bmatrix}, \tag{17}$$

where $\Sigma\epsilon\cos = \epsilon_1\cos\phi_1 + \epsilon_2\cos\phi_2$; $\Sigma\epsilon\sin = \epsilon_1\sin\phi_1 + \epsilon_2\sin\phi_2$; $\Delta\epsilon\cos = \epsilon_1\cos\phi_1 - \epsilon_2\cos\phi_2$; $\Delta\epsilon\sin = \epsilon_1\sin\phi_1 - \epsilon_2\sin\phi_2$. The imperfections in a good feed are completely specified by four independent parameters.

The central 4-element submatrix is a rotation matrix that represents an error in position angle of linear polarization (its diagonal elements are unity because of our first-order expansion in ϵ); in-phase mutual voltage coupling between the two probes causes an apparent rotation. Its off-diagonal elements $\Delta\epsilon\cos$ are impossible to measure without calibration sources whose position angles are accurately known. Moreover, a small rotation can also occur because of mechanical inaccuracy in mounting the feed. In practice, these problems make it impossible to separate this factor, so one might as well assume it is equal to zero and make appropriate adjustments to the position angle *ex post facto*.

Assuming $\Delta\epsilon\cos = 0$ is consistent with a more stringent assumption, namely that $\epsilon \equiv \epsilon_1 = \epsilon_2$ and $\phi \equiv \phi_1 = \phi_2$ so that, also, we have $\Delta\epsilon\sin = 0$. Physically, this is equivalent to assuming that the coupling in a linear feed arises from the two probes being not quite orthogonal and that the coupling in each has the same relative phase. In other words, it makes the correlated output non-orthogonal.

This assumption might seem to be too restrictive because, by requiring $\phi_1 = \phi_2$, it eliminates the possibility of imperfections inducing a change in the ellipticity of the polarization. However, it leads to no loss in generality of our treatment, because the out-of-phase coupling case is included in the feed matrix $\mathbf{M_F}$. Our restricted case of an imperfect feed is described by just two parameters:

$$\mathbf{M_{IFr}} = \begin{bmatrix} 1 & 0 & 2\epsilon\cos\phi & 2\epsilon\sin\phi \\ 0 & 1 & 0 & 0 \\ 2\epsilon\cos\phi & 0 & 1 & 0 \\ 2\epsilon\sin\phi & 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

3.5. The amplifiers

The two polarization channels go through different amplifier chains. Suppose these have *voltage* gain (g_A, g_B), *power* gain $(G_A, G_B) = (g_A^2, g_B^2)$, and phase delays (ψ_A, ψ_B). The Jones matrix is

$$\begin{bmatrix} E_{A,out} \\ E_{B,out} \end{bmatrix} = \begin{bmatrix} g_A e^{i\psi_A} & 0 \\ 0 & g_B e^{i\psi_B} \end{bmatrix} \begin{bmatrix} E_{A,in} \\ E_{B,in} \end{bmatrix} \tag{19}$$

In practice, the amplifier gains and phases are calibrated with a correlated noise source (the "cal"). Thus, our amplifier gains (G_A, G_B) have nothing to do with the actual amplifier gains. Rather, they represent the gains as calibrated by specified cal intensities, one for each channel. If the *sum* of the specified cal intensities is perfectly correct, then the absolute intensity calibration of the instrument is correct for an unpolarized source (Stokes I is correctly measured in absolute units). In our treatment, $G_A + G_B = 2$ by necessity because we deal with fractional polarizations.

If the *ratio* of the cal intensities is correct, then the difference between the two polarization channels is zero for an unpolarized source. This happy circumstance does generally not obtain. However, the relative cal intensities are known fairly well, which allows us to assume $\Delta G \equiv G_A - G_B \ll 1$ and to carry $g_A g_B$ to first order only, meaning we take $g_A g_B = 1$. With this first-order approximation, we have

$$\mathbf{M_A} = \begin{bmatrix} 1 & \frac{\Delta G}{2} & 0 & 0 \\ \frac{\Delta G}{2} & 1 & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{bmatrix} . \tag{20}$$

The incorrect relative cal amplitudes produce coupling of Stokes I into Q through the nonzero m_{QI}; the difference between the relative cal and sky phases produces a transfer of power between the two correlated outputs, as we now discuss.

The difference between the amplifier phases is also referred to the cal. Thus a phase difference $\psi = \psi_A - \psi_B$ represents the phase difference that exists between a linearly polarized astronomical source and the cal and has nothing to do with the amplifier chains. The behavior of this phase difference depends on the native polarization of the feed.

For a perfect native dual *linear* feed, the phase of a linearly polarized astronomical source, modulo 180°, is independent of PA_{az} because the linearly polarized dipoles have no relative phase difference ($\chi = 0$); the phase changes by 180° when the measured U changes sign as PA_{az} changes. Thus, ψ modulo 180° is independent of PA_{az}. The desirable case $\psi = 0$ means that $(E_A\overline{E_B} + \overline{E_B}E_A)$ contains pure Stokes U and $(E_A\overline{E_B} - \overline{E_B}E_A)$ pure Stokes V.

For a perfect native dual *circular* feed, the phase of a linearly polarized astronomical source (and therefore ψ) rotates as $\pm 2PA_{az}$ (see section 5.2 and equation 30b). At $PA_{az} = 0$, the condition $\psi = 0$ produces the correctly defined Stokes parameters in equation 14, $(E_A\overline{E_B} + \overline{E_B}E_A) = U_{sky}$

and $(E_A\overline{E_B} - \overline{E_B}E_A) = -Q_{sky}$. If $\psi \neq 0$, then the correctly defined Stokes parameters occur at $PA_{az} = \pm\frac{\psi}{2}$.

3.6. The correlator outputs

Our measured quantities are four time averaged voltage products from the digital correlator: from autocorrelation, (E_AE_A, E_BE_B); from crosscorrelation, (E_AE_B, E_BE_A). In these products we consider the second quantity to be delayed relative to the first. Each correlation function has N channels of delay. We Fourier transform (FT) these quantities to obtain spectra.

The autocorrelation functions are symmetric and thus their FT's are real, with no imaginary components. We combine the two measured crosscorrelation functions into a single one with $2N$ channels; it has both negative and positive delays and is generally not symmetric, so its FT is complex. We combine these Fourier transforms as in equation 3:

$$\begin{bmatrix} APB \\ AMB \\ AB \\ BA \end{bmatrix} = \begin{bmatrix} FT(E_AE_A) + FT(E_BE_B) \\ FT(E_AE_A) - FT(E_BE_B) \\ 2\ \mathrm{Re}[FT(E_AE_B)] \\ 2\ \mathrm{Im}[FT(E_AE_B)] \end{bmatrix} . \tag{21}$$

If we have an ideal native dual linear feed and a perfect receiver, then $(APB, AMB, AB, BA) = (I, Q, U, V)_{sky}$; for native circular, $(APB, AMB, AB, BA) = (I, V, U, -Q)_{sky}$.

4. THE SINGLE MATRIX FOR THE RADIOASTRONOMICAL RECEIVER

4.1. The general case with $\chi = 90°$

The observing system consists of several distinct elements, each with its own Mueller matrix. The matrix for the whole system is the product of all of them. Matrices are not commutative, so we must be careful with the order of multiplication.

We express the Jones vector of the incoming radiation in linear polarization. The radiation first encounters the feed, producing Stokes parameters as specified by $\mathbf{M_{Fr}}$ in equation 15. Next it suffers the restricted set of imperfections associated with $\mathbf{M_{IFr}}$ (equation 18). Finally it proceeds through the amplifier chains, undergoing $\mathbf{M_A}$ (equation 20). The product of these matrices, in this order ($\mathbf{M_{TOT}} = \mathbf{M_A} \cdot \mathbf{M_{IFr}} \cdot \mathbf{M_{Fr}}$), produces the vector that the correlator sees, which we denote by **COR**. In calculating **COR**, we ignore second order terms in the imperfection amplitudes $(\epsilon, \Delta G)$ but, of course, retain all orders in their their phases (ϕ, ψ) and also in the feed parameter α. This gives

$$\mathbf{M_{TOT}} = \begin{bmatrix} 1 & (-2\epsilon\sin\phi\sin 2\alpha + \frac{\Delta G}{2}\cos 2\alpha) & 2\epsilon\cos\phi & (2\epsilon\sin\phi\cos 2\alpha + \frac{\Delta G}{2}\sin 2\alpha) \\ \frac{\Delta G}{2} & \cos 2\alpha & 0 & \sin 2\alpha \\ 2\epsilon\cos(\phi+\psi) & \sin 2\alpha\sin\psi & \cos\psi & -\cos 2\alpha\sin\psi \\ 2\epsilon\sin(\phi+\psi) & -\sin 2\alpha\cos\psi & \sin\psi & \cos 2\alpha\cos\psi \end{bmatrix} . \quad (22)$$

The terms in the top row make $I \neq 1$ for a polarized source. If one derives fractional polarization, for example $\frac{Q}{I}$, then it will be in error by amounts comparable to $[(\epsilon, \Delta G) \times (Q, U, V)]$. For the weakly polarized sources we use as calibrators, these products are second order and therefore are of no concern.

However, for a strongly polarized source such as a pulsar, these terms are first order. This can be particularly serious for timing, because polarization variations across the pulse will produce errors in the pulse shape. These effects can be eliminated by correcting for the Mueller matrix.

4.2. Two important special cases

Commonly, feeds are intended to be either pure linear or circular. For these two important cases we have the following, expanded to first order:

(1) A dual-linear feed with a slight elliptical component, meaning that $[\alpha = (0^\circ, 90^\circ) + \delta\alpha]$ with $\delta\alpha \ll 1$. We ignore second order products involving $\delta\alpha$:

$$\mathbf{M_{TOT,lin}} = \begin{bmatrix} 1 & \pm\frac{\Delta G}{2} & 2\epsilon\cos\phi & 2\epsilon\sin\phi \\ \frac{\Delta G}{2} & \pm 1 & 0 & \pm 2\delta\alpha \\ 2\epsilon\cos(\phi+\psi) & \pm 2\delta\alpha\sin\psi & \cos\psi & \mp\sin\psi \\ 2\epsilon\sin(\phi+\psi) & \mp 2\delta\alpha\cos\psi & \sin\psi & \pm\cos\psi \end{bmatrix} . \quad (23)$$

For terms with two signs, the top sign is for the 0° case and the bottom for the 90° case.

(2) As above, but for a dual-circular feed with $[\alpha = (45^\circ, 135^\circ) + \delta\alpha]$:

$$\mathbf{M_{TOT,circ}} = \begin{bmatrix} 1 & \mp 2\epsilon\sin\phi & 2\epsilon\cos\phi & \pm\frac{\Delta G}{2} \\ \frac{\Delta G}{2} & \mp 2\delta\alpha & 0 & \pm 1 \\ 2\epsilon\cos(\phi+\psi) & \pm\sin\psi & \cos\psi & \pm 2\delta\alpha\sin\psi \\ 2\epsilon\sin(\phi+\psi) & \mp\cos\psi & \sin\psi & \mp 2\delta\alpha\cos\psi \end{bmatrix} . \quad (24)$$

Recall that in the circular case, the order of the Stokes parameters in the output vector is permuted: $[I, V, U, -Q]$. As with the imperfect linear, the imperfections in $\mathbf{M_{IFr}}$ produce coupling between Stokes I and $(V, U, -Q)$, represented by the nonzero elements in the left column. The order of the terms in the column is independent of the feed polarization, but the Stokes parameters are not, so the imperfections produce different effects for the two types of feed.

5. EVALUATING THE PARAMETERS IN THE MATRIX

We evaluate the parameters in equation 22 using observations of a polarized source tracked over a wide range of position angle PA_{az}. The source is described by $\mathbf{S_{src}}$ and the Mueller matrix for the radiation entering the feed by $\mathbf{M_{sky}}$, both described in section 3.1. The full Mueller matrix is $\mathbf{M_{TOT}} \cdot \mathbf{M_{sky}}$. The product of this matrix with $\mathbf{S_{src}}$ results in a set of of four equations, one for each element of the observed **COR** vector. Recalling that we define the source Stokes parameters as fractional (so that $I_{src} = 1$) and that we assume $V_{src} = 0$, they are expressed by the four equations embodied in

$$\begin{bmatrix} APB \\ AMB \\ AB \\ BA \end{bmatrix} = \mathbf{M_{TOT}} \cdot \mathbf{M_{SKY}} \cdot \begin{bmatrix} 1 \\ Q_{src} \\ U_{src} \\ 0 \end{bmatrix} . \tag{25}$$

In practice, we cannot reliably measure the PA_{az} dependence of *APB*. *APB* is approximately equal to the total intensity, and it is rendered inaccurate by small gain errors. Thus, in practice we use fractional correlator outputs. We define

$$\mathbf{COR}' = \frac{\mathbf{COR}}{APB} \tag{26}$$

and consider only the last three equations in equation 25:

$$\begin{bmatrix} AMB' \\ AB' \\ BA' \end{bmatrix} = \mathbf{M_{TOT}} \cdot \mathbf{M_{SKY}} \cdot \begin{bmatrix} Q_{src} \\ U_{src} \\ 0 \end{bmatrix} . \tag{27}$$

Note that the division by *APB* produces errors in the other elements of $\mathbf{COR}'$, but these errors are second order because they are products of ΔG and/or ϵ with quantities such as Q_{src} that are already first order. Our whole treatment neglects second order products, so we can neglect these errors.

Multiplying out equation 27, we obtain three equations of the form

$$AMB = A_{AMB'} + B_{AMB'} \cos 2PA_{az} + C_{AMB'} \sin 2PA_{az} \ , \tag{28}$$

where the coefficients (A, B, C) are complicated function of the 5 parameters $(\alpha, \epsilon, \phi, \Delta G, \psi)$ and, also, the two source Stokes parameters (Q_{src}, U_{src}) which are not known *ab initio*, so we have 7 unknown parameters. We have 9 measured quantities, three (A, B, C) for each correlator output; these are derived from least squares fits of (AMB', AB', BA') to PA_{az}. We use nonlinear least squares fitting to solve for the 7 unknown parameters. In practice, we use numerical techniques to obtain the relevant derivatives. An alternative fitting technique that is useful when combining the results from N_{src} different sources into a grand average is to express the coefficients $A_{AMB'}$, etc., in terms of the $(5 + 2N_{src})$ unknown coefficients, and to lump all observations of all sources together in one grand nonlinear least squares fit.

Nonlinear least square fitting is often plagued by multiple minima, and the present case is no exception when the polarization is nearly pure linear or circular. To discuss these cases we temporarily assume $(\epsilon, \Delta G) = 0$, which makes the mathematics more transparent.

5.1. The nearly linear case

Equation 25 becomes

$$\begin{bmatrix} Q_{out} \\ U_{out} \\ V_{out} \end{bmatrix} \approx \begin{bmatrix} AMB' \\ AB' \\ BA' \end{bmatrix} = \begin{bmatrix} \pm 1 & 0 \\ \pm 2\delta\alpha \sin\psi & \cos\psi \\ \mp 2\delta\alpha \cos\psi & \sin\psi \end{bmatrix} \begin{bmatrix} \cos 2PA_{az} & \sin 2PA_{az} \\ -\sin 2PA_{az} & \cos 2PA_{az} \end{bmatrix} \begin{bmatrix} Q_{src} \\ U_{src} \end{bmatrix} \tag{29a}$$

which, when multiplied out, becomes

$$\begin{bmatrix} AMB' \\ AB' \\ BA' \end{bmatrix} = \begin{bmatrix} \pm\cos 2PA_{src} & \pm\sin 2PA_{src} \\ \cos\psi \sin 2PA_{src} \pm 2\delta\alpha \sin\psi \cos 2PA_{src} & -\cos\psi \cos 2PA_{src} \pm 2\delta\alpha \sin\psi \sin 2PA_{src} \\ \sin\psi \sin 2PA_{src} \mp 2\delta\alpha \cos\psi \cos 2PA_{src} & -\sin\psi \cos 2PA_{src} \mp 2\delta\alpha \cos\psi \sin 2PA_{src} \end{bmatrix} \begin{bmatrix} P_{src} \cos 2PA_{az} \\ P_{src} \sin 2PA_{az} \end{bmatrix} . \tag{29b}$$

For $\delta\alpha = 0$ we have

$$\begin{bmatrix} AMB' \\ AB' \\ BA' \end{bmatrix} = \begin{bmatrix} \pm\cos 2PA_{src} & \pm\sin 2PA_{src} \\ \cos\psi \sin 2PA_{src} & -\cos\psi \cos 2PA_{src} \\ \sin\psi \sin 2PA_{src} & -\sin\psi \cos 2PA_{src} \end{bmatrix} \begin{bmatrix} P_{src} \cos 2PA_{az} \\ P_{src} \sin 2PA_{az} \end{bmatrix} . \tag{29c}$$

This shows that, for $\delta\alpha \to 0$, one cannot distinguish between the two cases $(\alpha, \psi) = (0°, \psi_0)$ and $(\alpha, \psi) = (90°, \psi_0 + 180°)$, because the two bottom rows change sign for $\psi_0 + 180°$. For these two cases, the signs of (Q, U) change, which is equivalent to rotating the feed by 90°. The physical interpretation is straightforward: $\alpha = 90°$ converts E_X to E_Y in equation 10, thus changing the sign of Q; changing ψ by 180° changes the sign of U; the combination is equivalent to rotating the feed by 90°.

In practice one must deal with this problem. For a conventional linear feed, loosely described as two E-field probes in a circular waveguide, the combination $(\alpha, \psi) = (90°, 180°)$ is physically unreasonable. However, for some feeds, in particular a turnstile junction operating in linear polarization, either possibility can occur and one must make the appropriate choice based on frequencies well away from where the feed is pure linear; for a turnstile we expect $\frac{d\alpha}{df} \approx const.$

5.2. The nearly circular case

Equation 25 is

$$\begin{bmatrix} V_{out} \\ U_{out} \\ -Q_{out} \end{bmatrix} \approx \begin{bmatrix} AMB \\ AB \\ BA \end{bmatrix} = \begin{bmatrix} \mp 2\delta\alpha & 0 \\ \pm \sin\psi & \cos\psi \\ \mp \cos\psi & \sin\psi \end{bmatrix} \begin{bmatrix} \cos 2PA_{az} & \sin 2PA_{az} \\ -\sin 2PA_{az} & \cos 2PA_{az} \end{bmatrix} \begin{bmatrix} Q_{src} \\ U_{src} \end{bmatrix} \tag{30a}$$

and becomes

$$\begin{bmatrix} AMB \\ AB \\ BA \end{bmatrix} = \begin{bmatrix} \mp 2\delta\alpha \cos 2PA_{src} & \mp 2\delta\alpha \sin 2PA_{src} \\ \sin(2PA_{src} \pm \psi) & -\cos(2PA_{src} \pm \psi) \\ \mp \cos(2PA_{src} \pm \psi) & \mp \sin(2PA_{src} \pm \psi) \end{bmatrix} \begin{bmatrix} P_{src} \cos 2PA_{az} \\ P_{src} \sin 2PA_{az} \end{bmatrix} . \tag{30b}$$

This shows that the angles $2PA_{src}$ and ψ are inextricably connected. We can determine only their sum (for $\alpha = 45°$) or their difference. It also shows that the two solutions $\alpha = (45°, -45°)$ are degenerate as $\delta\alpha \to 0$, because the bottom row changes sign for these two cases (thus flipping the derived sign of Q_{src} and thereby rotating the derived PA_{src} by 90°). The physical interpretation of this degeneracy is straightforward: for a pure circular feed, the phase of a linearly polarized source rotates with $2PA_{az}$ and its absolute value depends both on the system phase ψ and the source position angle PA_{src}.

In practice, one can only deal with this problem by having additional information, namely knowing either ψ or PA_{src}. For the case of turnstile junctions, which are narrow band devices, one can determine ψ at frequencies well away from that where the feed is pure circular and interpolate. Then, during the fit, this value of ψ should be fixed. For a wideband circular feed, there is no substitute for an independent calibration of the linearly polarized position angle, either with a test radiator or with a source of known polarization.

5.3. Commentary

(1) ϵ is the quadrature sum of the PA_{az}-independent portions of the two correlated outputs (AB, BA). This power is distributed between those outputs according to $(\phi + \psi)$. In the near-linear case, ψ can change by 180° by changing the choice for α, and this also produces a 180° change in ϕ.

(2) Consider a high-quality standard linearly polarized feed that has a correlated cal connected by equal-length cables. Such a feed has $\alpha \approx 0°$ and the equal-length cables mean that $\psi \approx 0°$. However, the solution yields $\psi \approx 180°$ if the sign of AMB is incorrect, which can easily happen if one interchanges cables carrying the two polarizations; this is equivalent to reversing the handedness of PA_{az} and PA_{src}.

(3) If one has a system without a correlated cal, then ψ is meaningless. ψ has contributions at r.f. (from the difference in cable and electrical lengths to the first mixer) and i.f. (from length differences after the mixer). Normally the latter is likely to dominate because the cable runs from the feed to the control room are long. For example, at Arecibo we found $\frac{d\psi}{df} \sim 0.1$ rad MHz^{-1}, roughly constant among different systems. This corresponds to an electrical length difference of ~ 5 m, most of which probably occurs along the pair of ~ 500 m optical fibers that carry the two channels from the feed to the control room.

(4) We have adopted the following procure for phase calibration. If there is a correlated cal then we measure ψ_{cal} and fit it to a constant plus a slope $\frac{d\psi}{df}$; we subtract this fit from the source phase and produce corrected versions of (AB, BA). Thus, the only component left in the correlated products is the difference between source and cal phase, which is the same as ψ in the equation 20.

If there is not a correlated cal, then we measure ψ_{src} and fit it to a constant plus a slope $\frac{d\psi}{df}$; we subtract the *slope* but not the constant from the source phase and produce corrected versions of (AB, BA). While most of this slope is in the system, this procedure also subtracts away any intrinsic slope caused by Faraday rotation.

6. APPLYING THE CORRECTION

One of the major reasons to determine the Mueller matrix elements is to apply it to observations and obtain true Stokes parameters. There are two steps to this process.

6.1. Applying $\mathbf{M_{TOT}}$ and $\mathbf{M_{sky}}$

The completely general form of equation 25 uses observed voltage products instead of fractional ones and does not force $V = 0$. Thus, to derive the source Stokes parameters from the data, we use

$$\begin{bmatrix} I_{src} \\ Q_{src} \\ U_{src} \\ V_{src} \end{bmatrix} = (\mathbf{M_{TOT}} \cdot \mathbf{M_{SKY}})^{-1} \cdot \begin{bmatrix} APB \\ AMB \\ AB \\ BA \end{bmatrix} . \tag{31}$$

6.2. Deriving true astronomical position angles

The position angle of the source polarization PA_{src} is defined relative to the local idiosyncrasies. For a dual linear feed these include the angle at which feed probes happen to be mounted and, also, which feed probe happens to be defined as A. For a dual circular feed this includes the phase angle at which the correlated cal happens to be injected and the angle at which the feed happens to be oriented.

Astronomers wish to express position angles in the conventional way, *viz.* with PA_{src} measured relative to the North Celestial Pole. There is also the possibility of its handedness, but this is taken care of automatically in the fitting process if the PA_{az} is correctly defined. To satisfy the astronomers' desire, we must apply a rotation matrix $\mathbf{M_{astron}}$, which looks like $\mathbf{M_{SKY}}$ in equation 9 with PA_{az} replaced by θ_{astron}. For a linearly polarized feed the angle θ_{astron} is the angle of the feed probes with respect to azimuth arm. There is a sign ambiguity that is best determined by empirical comparison with known astronomical position angles. At low frequencies, one must include the effects of terrestrial ionospheric Faraday rotation, which is time variable.

7. SAMPLE RESULTS

Here we present sample results for the two Arecibo L-band receivers (L-band is the frequency range surrounding the 21-cm line). For each observing session the digital correlator was split into four 25-MHz chunks centered at different frequencies. The L-band Wide receiver (LBW) is a very wide-bandwidth dual linearly polarized feed. The L-band Narrow receiver (LBN) is a turnstile junction whose polarization state changes from dual linear to dual circular over a frequency range $\sim$ 100 MHz. First, however, we reiterate the definitions of the parameters.

7.1. Reiteration of parameter definitions

$\boldsymbol{\Delta G}$ is the error in relative intensity calibration of the two polarization channels. It results from an error in the relative cal values (T_{calA}, T_{calB}). Our expansion currently takes terms in ΔG to first order only, so if the relative cal intensities are significantly incorrect then the other parameters will be affected.

The relative cal values should be modified to make $\Delta G = 0$, keeping their sum the same. To accomplish this, make $T_{calA,modified} = T_{calA}\left(1 - \frac{\Delta G}{2}\right)$ and $T_{calB,modified} = T_{calB}\left(1 + \frac{\Delta G}{2}\right)$.

$\boldsymbol{\psi}$ is the phase difference between the cal and the incoming radiation from the sky; see the discussion following equation 20. It redistributes power between (U, V) for a dual linear feed and between (Q, U) for a dual circular feed (equations 23 and 24).

$\boldsymbol{\alpha}$ is a measure of the voltage ratio of the polarization ellipse produced when the feed observes pure linear polarization. Generally, the electric vector traces an ellipse with time; $\tan\alpha$ is the ratio of major and minor axes of the voltage ellipse. Thus, $\tan^2\alpha$ is the ratio of the powers. If a source having fractional linear polarization $P_{src} = \sqrt{Q_{src}^2 + U_{src}^2}$ is observed with a native circular feed that has $\alpha = \frac{\pi}{4} + \delta\alpha$, with $\delta\alpha \ll 1$, then the measured Stokes V will change with $2PA_{az}$ and have peak-to-peak amplitude $4\delta\alpha$.

$\boldsymbol{\chi}$ is the relative phase of the two voltages specified by α. Our analysis assumes $\chi = 90°$; this incurs no loss of generality, as explained in sections 3.3 and 3.4.

$\boldsymbol{\epsilon}$ is a measure of imperfection of the feed in producing nonorthogonal polarizations (false correlations) in the two correlated outputs. Our expansion takes ϵ to first order only. The only astronomical effect of nonzero ϵ is to contaminate the polarized Stokes parameters (Q, U, V) by coupling Stokes I into them at level $\sim 2\epsilon$; the exact coupling depends on the other parameters. For weakly polarized sources, this produces false polarization; for strongly polarized sources such as pulsars, it also produces incorrect Stokes I.

$\boldsymbol{\phi}$ is the phase angle at which the voltage coupling ϵ occurs. It works with ϵ to couple I with (Q, U, V).

$\boldsymbol{\theta_{astron}}$ is the angle by which the derived position angles must be rotated to conform with the conventional astronomical definition.

7.2. Results for LBW

LBW is a very wide-band feed with native linear polarization. It has some problems with resonances. Apart from these, the parameters are nearly ideal and frequency-independent from 1175 to 1680 MHz: $\alpha \sim 0.25°$ and $\epsilon \sim 0.0015$. Care was taken by the receiver engineer to equalize the length of the two cables for the correlated cal; as a result, $\psi \sim 4.6°$.

Figure 1 exhibits observational results for two sources obtained during two Mueller-matrix calibration observing sessions. The digital correlator was split into four 25-MHz chunks, with duplicate coverage at 1415 MHz on the two days; this provides seven different center frequencies. The change of position angle from Faraday rotation is obvious over the broad band covered by the data, and even within a single spectrum. Regarding fractional polarization, the decrease toward low frequency for B0017+154 is not surprising; this is caused by Faraday depolarization. However, that of B1634+269 reaches a minimum near 1400 MHz, and this behavior is somewhat unusual. We believe that our measurements are correct and that this observed behavior is real.

7.3. Results for LBN

LBN is a turnstile system without a correlated cal and is commonly used over a large frequency range. Turnstile junctions are narrow-band devices for two reasons: one, the polarization response is defined by physical path length differences in waveguide, and these length differences correspond to incorrect phase differences as one moves away from the design frequency; two, unwanted reflections within the junction are eliminated by a tuning structure, and this is narrow-band. The solid lines in Figure 2 exhibit the frequency dependence of ΔG, α, ϵ, and ϕ for the 25 MHz bands centered at four frequencies within the range commonly used with this feed. These particular data were derived from the source B0017+154; we obtained data for two additional sources, and the results agree well. The dashed lines are our adopted analytic expressions for the frequency dependence of the parameters.

Near 1415 MHz the polarization is dual circular; the absence of a correlated cal means that the parameter ψ has no meaning and, moreover, we cannot measure the position angle of linear polarization. The dependence of α on frequency is close to linear, which is what's expected for a turnstile junction. The variation of ϵ is remarkably complicated, probably because of resonances in the tuning structure, and we do not have sufficient frequency coverage to characterize it. The scatter in ϕ for the 1375 MHz spectrum simply reflects the uncertainty in the angle, which is large because ϵ is small. Of course, ΔG simply reflects inaccurate relative cal values and not the properties of the turnstile itself.

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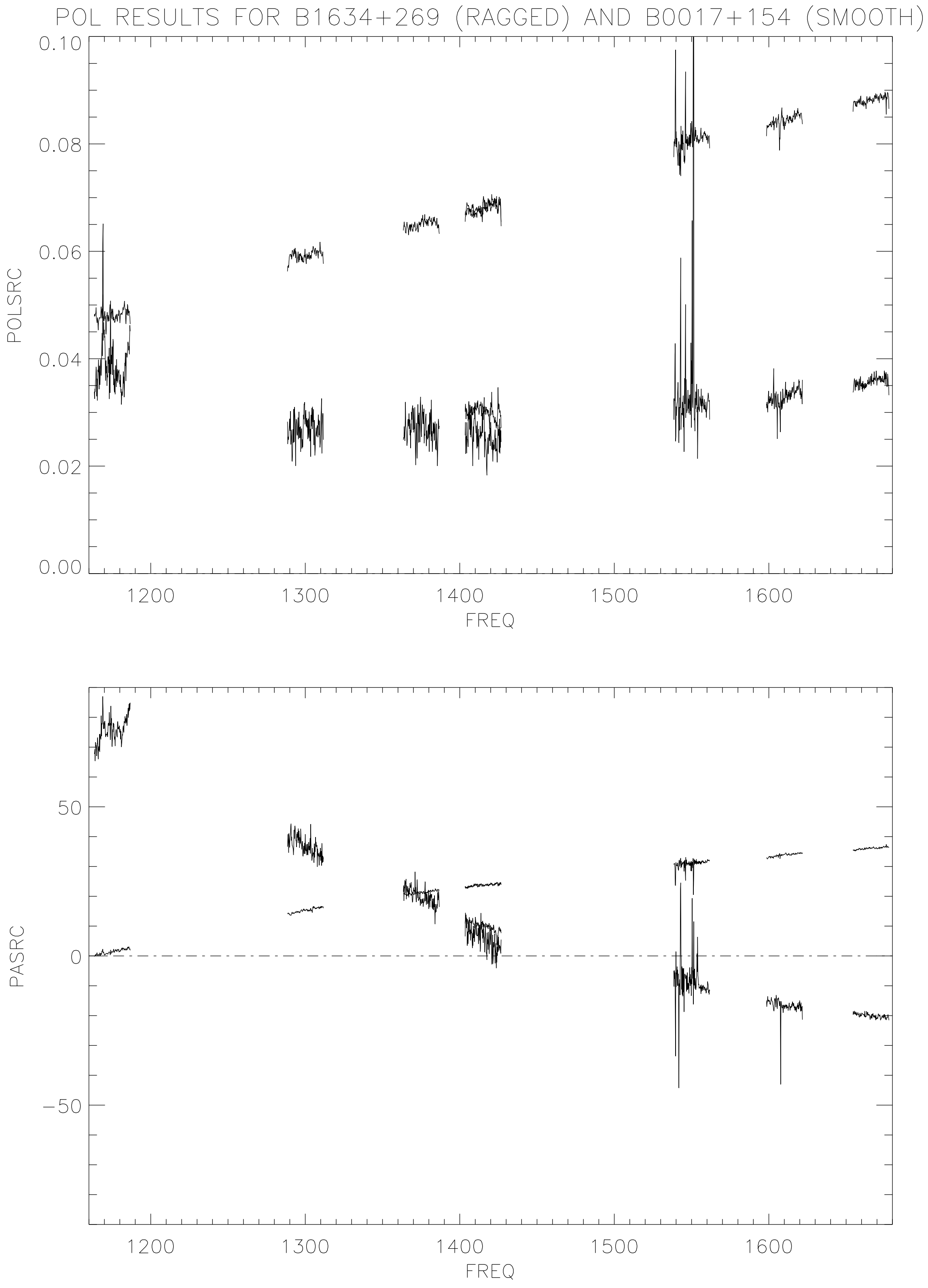
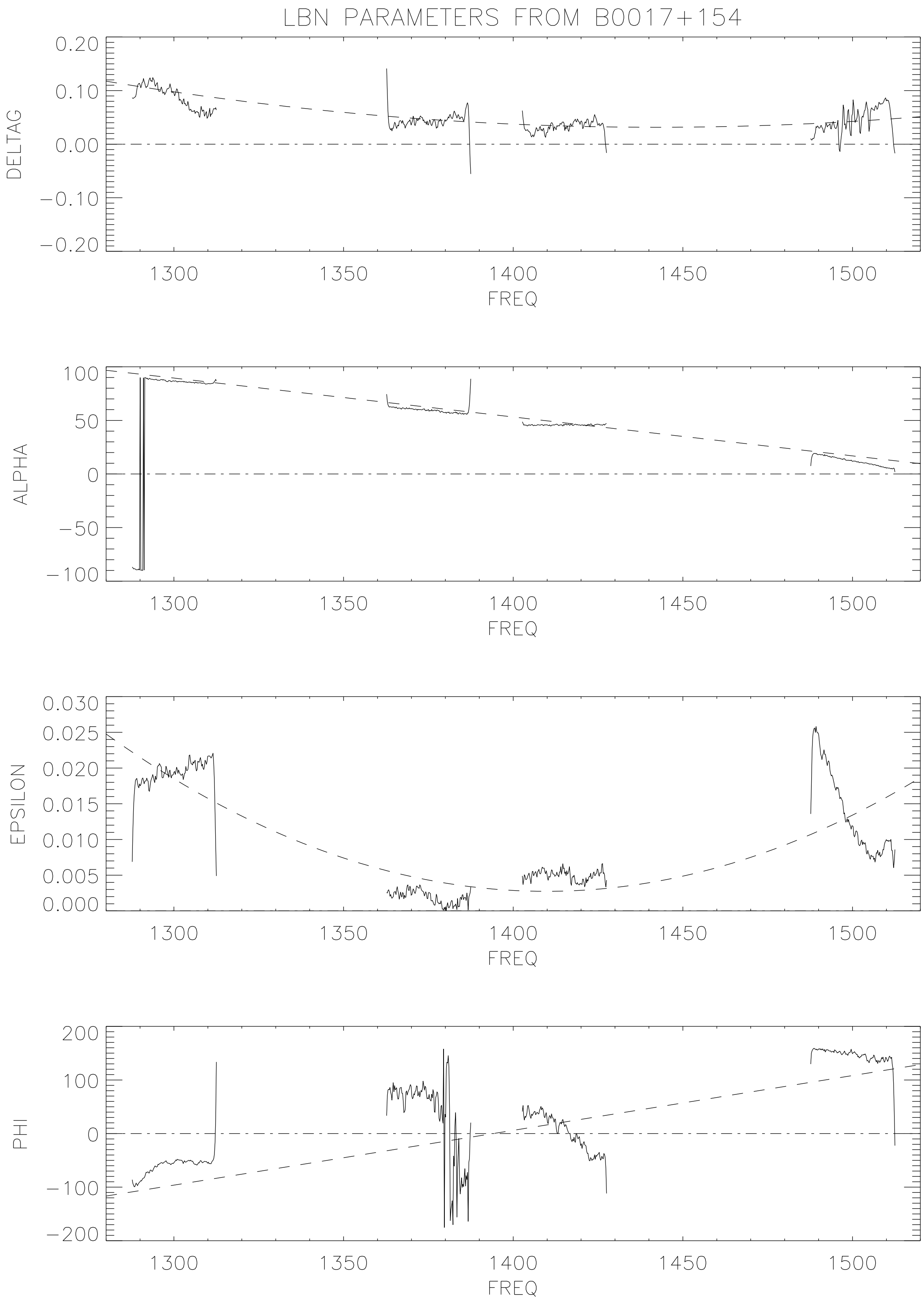
8. FIGURE CAPTIONS

Fig. 1.— Polarizations of B1634+269 and B0017+154, derived from the calibration measurements for LBW. Position angles have not been rotated by θ_{astron}.

Fig. 2.— Mueller matrix parameters versus frequency for LBN, together with our adopted analytic approximations. ψ is meaningless and not shown because this receiver does not have a correlated cal.


POL RESULTS FOR B1634+269 (RAGGED) AND B0017+154 (SMOOTH)
POLSRC
0.00
0.02
0.04
0.06
0.08
0.10
1200
1300
1400
1500
1600
FREQ
PASRC
50
0
−50
1200
1300
1400
1500
1600
FREQ


LBN PARAMETERS FROM B0017+154
DELTAG
ALPHA
EPSILON
PHI
FREQ
1300
1350
1400
1450
1500